PERGAMON

International Journal of Solids and Structures 36 (1999) 5141–5176

INTERNATIONAL JOURNAL OF SOLIDS and STRUCTURES

www.elsevier.com/locate/ijsolstr

A two-dimensional model for linear elastic thick shells

R. Nzengwa*, B. H. Tagne Simo

Civil Engineering Department, Ecole Nationale Supérieur Polytechnique de Yaoundé I, B.P. 8390, Yaounde, Cameroon

Received 15 October 1996; in revised form 2 April 1998

Abstract

In this paper, a two-dimensional model for linear elastic thick shells is deduced from the three-dimensional problem of a shell thickness 2ε, $\varepsilon > 0$. From different scalings on the tangent and normal components of the displacement u^{ε} as widely used in recent works, the limit displacement appears to be Kirchhoff–Love displacement of a different type. It contains additional terms to those found in the Reissner–Mindlin model and satisfies more general equations containing the classical terms found in the literature and some new terms related to the third fundamental form. Such terms could not be well handled in the usual framework. Shear stresses across the thickness are also computed. This model appears to be appropriate to handle stiffened shells which, in fact, cannot be considered uniformly as shallow shells. As a by-product it also lays the mathematical background to justify the Reissner–Mindlin model for plates and will probably contribute to a better understanding of the locking phenomenon encountered in computational mechanics. © 1999 Elsevier Science Ltd. All rights reserved.

Nomenclature

$a_{..}$ $b_{..}$ $c_{..}$	first, second and third fundamental form of the surface S
A_{ε}^{ijkl}, A^{ijkl}	contravariant components of the three-dimensional elasticity tensor of the shell $\bar{\Omega}^{\varepsilon}$ and $\bar{\Omega}$, respectively
$\bar{A}^{\alpha\beta\gamma\delta}$	contravariant components of the best-first order two-dimensional elasticity tensor defined on the middle surface S
$(AB)^{ij} = A^{ijkl}B_{kl}$	if $A = (A^{ijkl}(x))$, $B = (B_{ij})$
$b_{\alpha\beta}$, b_{β}^{α}	covariant and mixed components of the curvature tensor $b_{..}$
$B:C = B^{ij}C_{ij}$	if $B = (B^{ij})$, $C = (C_{ij})$
$\det(a_{..})$, $\det(g_{..})$	determinant of the metric tensors $(a_{..})$ and $(g_{..})$
$e_{..}$	covariant components of the linearized strain tensor of the middle surface S
$f \cdot v = f^i v_i$	if $f = (f^i)$, $v = (v_i)$
(f_{ε}^i), (f^i)	contravariant components of the applied body forces in Ω^{ε} and Ω, respectively

* Corresponding author. Fax: 00 237 23 18 41; e-mail: rnzengwa@polytech.uninet.cm; or rnzengwa@hotmail.com

PII: S0020-7683(98)00165-6

$g^{\varepsilon}_{..}$, $g^{..}_{\varepsilon}$ and $g_{..}$, $g^{..}$	covariant and contravariant components of the metric tensor of the shell $\bar{\Omega}^{\varepsilon}$ and the shell $\bar{\Omega}$, respectively
(g^{i}_{ε}), (g^{i})	contravariant components of the applied surface forces on the borders $\partial\Omega^{\varepsilon}$ and $\partial\Omega$, respectively
$\bar{H}$, $\bar{K}$	mean curvature and Gauss curvature of the shell, respectively
IR^{3}	Euclidean space
$k_{..}$	covariant components of the linearized change of curvature tensor of the surface S
m^{s}	flexural moment density on the part γ_1 of the border ∂S
m^{α}	contravariant components in the tangent plane of applied moment on the part γ_1 of the border ∂S
m^{v}	torsional moment density on the part γ_1 of the border ∂S
$M^{\alpha\beta}_{\varepsilon}$, $\bar{M}^{\alpha\beta}_{\varepsilon}$, $\bar{\bar{M}}^{\alpha\beta}_{\varepsilon}$	contravariant components of the flexural moments (or bending moments) stress tensors M_{ε}, $\bar{M}_{\varepsilon}$ and $\bar{\bar{M}}_{\varepsilon}$, respectively
$N^{\alpha\beta}_{\varepsilon}$, $\bar{N}^{\alpha\beta}_{\varepsilon}$	contravariant components of the resultant (or membrane) stress tensors N_{ε} and $\bar{N}_{\varepsilon}$, respectively
$\bar{N}^{\alpha\beta\gamma\delta}_{n}$, $n = 0, 1, 2, 3, 4$	contravariant components of the full two-dimensional elasticity tensor defined on the middle surface S
p^{α}, p^{3}	contravariant components of the two-dimensional applied surface density loads
q^{α}, q^{3}	contravariant components of the two-dimensional applied border density loads
$Q_{..}$	covariant components of the linearized change in the third fundamental form of S, or change of Gauss curvature tensor of the surface S
R_{α}	principal radii of the surface S
R_{ijkl}	Riemann–Christoffel tensor of the derivation $\vert$
$\bar{R}_{\alpha\beta\gamma\delta}$	Riemann–Christoffel tensor of the derivation ∇
S	reference configuration of the middle surface of the shell
$u^{\varepsilon} = (u^{\varepsilon}_{i})$, $u = (u_i)$, $v^{\varepsilon} = (v^{\varepsilon}_{i})$ or $v = (v_{\mathrm{i}})$	three-dimensional displacement fields and vector fields defined on $\bar{\Omega}^{\varepsilon}$ and $\bar{\Omega}$, respectively
$\bar{u} = (\bar{u}_i)$, $\bar{v} = (\bar{v}_i)$	two-dimensional displacement and vector fields defined on the middle surface S
$x^{\varepsilon} = (x^{x}_{i})$ and $x = (x_i)$	are generic point in the sets $\bar{\Omega}^{\varepsilon}$ and $\bar{\Omega}$, respectively.

Greek symbols

Γ^{k}_{ij}, $\bar{\Gamma}^{\gamma}_{\alpha\beta}$	Christoffel symbols defined on Ω^{ε} and S, respectively
δ^{α}_{β}, δ^{i}_{j}, δ^{ij}, δ_{ij}	Kronecker's symbols
$\epsilon_{..}(u^{\varepsilon})$, $\epsilon^{..}(u^{\varepsilon})$	three-dimensional covariant and contravariant components of the linearized strain tensor associated to the displacement field u^{ε}, respectively
ε	variable, defines the half-thickness of the shell
$\sigma^{..}_{\varepsilon}$, $\sigma^{..}$	the three-dimensional contravariant components of the stress linearized tensor of the shell $\bar{\Omega}^{\varepsilon}$ and $\bar{\Omega}$, respectively
$\bar{\Omega}^{\varepsilon}$, $\bar{\Omega}$	closures of the sets Ω^{ε} and Ω and are the reference configurations of the shell of thickness 2ε and 2, respectively

∇_α	covariant derivation on S
$\nabla\varphi = (\partial_i\varphi_j)$	gradient of a mapping $\varphi : \Omega \subset IR^3 \to IR^3$.

Other symbols

∂A	border of the subset A in the Euclidean space.
$\cdot$	Euclidean scalar product
$:$	matrix scalar product
$\times$	Euclidean vector product
$\lvert\cdot\rvert$	Euclidean norm
$\vert$	covariant derivation on Ω^ε or Ω.

1. Introduction

Models for thin (or shallow) shells have been widely analyzed since the works of John (1965), Koiter (1970), who laid, modern foundations, to recent works due to Ciarlet and Miara (1992), Ciarlet and Lods (1994, 1996), Blouza and Le Dret (1995) and other authors (see references) who have reviewed and justified the linear models including many practical aspects on the loads and geometry of the midsurface. The efficiency of the numerical computation of the models so obtained depends crucially on the ratio $\chi = h/R$, where h is half the thickness and R the absolute value of the minimum radius of the midsurface or on some small dimensionless parameter (see e.g. Brezzi and Fortin, 1986). Terms proportional to χ are not found in these models (see Bamberger, 1975). Such terms will involve the third fundamental form of the midsurface which disappears in the limit analysis even in recent works. In thick shells, energy related to terms proportional to higher orders (greater or equal to one) of χ can be important and such terms may improve not only the numerical methods, but also, provide much more information on the shear stresses usually important in this case. Elastic energy stored in these terms may improve elastic behaviour in elasto-plastic analysis and even mode shapes in dynamic analysis.

In this paper, a two-dimensional model for linear elastic thick shells is deduced from the three-dimensional problem of a shell of thickness 2ε, $\varepsilon > 0$, under the hypothesis that the Lamé constants Λ^ε, G^ε of the initial shell vary as ε^{-3}. Different scalings are performed on the tangent and normal components of the displacement u^ε as widely used in recent works. However, there is a main difference in the scaling procedure. Let Ω^ε be the domain occupied by the shell and (g_i^ε), $(g^{i,\varepsilon})$ its covariant and contravariant basis, respectively. Let Ω be the homologous scaled domain. In recent works (see e.g. Ciarlet and Miara, 1992; Ciarlet and Lods, 1994, 1996), (g_i^ε) and $(g^{i,\varepsilon})$ are assumed to constitute the covariant and contravariant basis of Ω. So Ω^ε and Ω have the same metric. All vectors and tensors in the fixed domain Ω, therefore, have their components expressed in basis depending on ε. Consequently metric systems in the scaled domain Ω are a priori approximated. In our setting, the exact covariant and contravariant basis in Ω are used. Moreover all tensors and vectors in the scaled or unscaled configurations are expressed in the midsurface basis which is fixed. This difference is very important and consequently, the displacement obtained in our limit analysis is more general. It contains additional terms to those found in the Reissner–Mindlin model and satisfies more general equations containing the classical terms found in the literature and some new terms related to the third fundamental form as already stated. Shear stresses along the thickness are computed by solving some differential equations slightly different from those found in thin plate theory (Destuynder, 1980). This model appears to be appropriate to handle stiffened shells as they cannot be considered uniformly as shallow shells. It also lays the mathematical background to justify Reissner–Mindlin model for plates and will probably contribute to a better understanding of the locking phenomenon encountered in computational mechanics.

This paper is organized as follows: in Section 1, in order to let this paper be self content, and for notational purposes, we present usual preliminaries in shell theory and functional spaces which shall not be modified in our approach. Some useful new relations related to this approach will be demonstrated. Most demonstrations found in the literature will be briefly referred to without further details. The unscaled and scaled three-dimensional problem will be analyzed in Section 2. Apart from the difference noted above, the scaling procedure is the same as in Ciarlet et al. (1989), Ciarlet and Miara (1992), Ciarlet and Lods (1994, 1996) and other authors except on the curvature tensor where we have introduced a crucial hypothesis not widely used. In Section 3, convergence results shall be established and shear stresses computed. Unlike in plates and thin shell classical theory, the limit problem appears to be the equilibrium equations of a shell with a non homogeneous modulus tensor which depends locally on the thickness and curvature. Terms appearing in this tensor are proportional to (χ^p), $p \geqslant 0$. The first-order term of the Taylor expansion of the modulus tensor also leads to a well defined homogeneous shell equation similar to that found in engineering literature with some additional terms. A detailed analysis of this first-order two-dimensional model for thick shells is given in Section 4. Some comments will be made in Section 5.

All vector spaces in IR^n, $n \geqslant 1$ will be Euclidean with an orthonormal basis whose scalar product will be denoted by $\cdot$ and vector product by $\times$. We shall also denote $v_{,\alpha} = \partial v/\partial x_\alpha = \partial_\alpha v$. All indices in Greek letters take their values in the set $\{1, 2\}$ while Latin indices range in $\{1, 2, 3\}$. The repeated index convention on summation will be adopted unless otherwise specified. All constants C used will be independent of the different variables unless otherwise specified. Further notations will be given in the text or in the glossary.

2. Preliminaries

Most of the results, based on the application of differential geometry on a surface can be found in Do Carmo (1976), Spivak (1975), Naghdi (1970), Green and Zerna (1968), Lelong and Ferrand (1963) and Morgenstern (1959).

2.1. Geometry and deformation of a surface

In what follows, S is a sufficiently regular bounded surface in IR^3 defined by a chart $\varphi\colon \bar{\omega} \subset IR^2 \rightarrow IR^3$, such that $\varphi(\bar{\omega}) = \bar{S}$, S has a boundary at least Lipschitz continuous, ω is an open bounded connected subset of IR^2; $\bar{S}$ and $\bar{\omega}$ denote the closure of S and ω, respectively. We shall assume that the mapping φ is of class C^3 though milder conditions could be used in defining the surface tensors (see Blouza and Le Dret, 1995). In fact it suffices that φ belongs to the space $W^{2,\infty}(\bar{\omega})$ which implies that S is C^1 (see Adams, 1975).

Let $\bar{x} = (x^1, x^2)$ be the coordinate system in S, then the covariant and contravariant tangent basis of S are defined by (a_α) and (a^α) where

$$a_\alpha = \partial_\alpha \varphi \quad \text{and} \quad a^\alpha \cdot a_\beta = \delta^\alpha_\beta.$$

Let $a_3(\bar{x}) = a_1 \times a_2/|a_1 \times a_2| = a^3(\bar{x})$, then a_1, a_2, a_3 or a^1, a^2, a^3 constitute two dual basis in IR^3. The covariant and contravariant metric tensors on S, also called the first fundamental form of S, are defined by

$$(a_{\alpha\beta}) = (a_\alpha \cdot a_\beta) = (a_\beta \cdot a_\alpha) = (a_{\beta\alpha}); \quad (a^{\alpha\beta}) = (a^\alpha \cdot a^\beta) = (a^\beta \cdot a^\alpha) = (a^{\beta\alpha}).$$

An area element is defined by: $\mathrm{d}S\sqrt{a}\,\mathrm{d}x^1\,\mathrm{d}x^2$; where $a = \det\,(a_{\alpha\beta})$. The curvature tensor on S also called the second fundamental form is defined by

$$b = (b_{\alpha\beta}) = (a_{\alpha,\beta} \cdot a_3) = (a_{\beta,\alpha} \cdot a_3) = -(a_{3,\alpha} \cdot a_\beta) = -(a_{3,\beta} \cdot a_\alpha).$$

This symmetric tensor is also defined by its mixed components: $b^\alpha_\beta = a^{\alpha\delta} b_{\delta\beta}$.

Let R_1 and R_2 be the eigenvalues of b (also called principal radii) then we shall denote the mean curvature and Gauss curvature, respectively, by (Do Carmo, 1976; Naghdi, 1970)

$$\bar{H} = \frac{1}{2}\left(\frac{1}{R_1} + \frac{1}{R_2}\right) = \frac{1}{2}\operatorname{tr} b = \frac{1}{2} b^\alpha_\alpha, \quad \bar{K} = \frac{1}{R_1}\frac{1}{R_2} = \det(b^\beta_\alpha).$$

The third fundamental form scarcely mentioned in the literature and defined by

$$c = (c_{\alpha\beta}) = (b^\lambda_\alpha b_{\lambda\beta}) = (b_{\alpha\lambda} b^\lambda_\beta) = (c_{\beta\alpha})$$

will be of paramount interest in our subsequent analysis.

A surface tensor $\bar{T}$ will either be expressed by its covariant, contravariant or mixed components $(\bar{T}^\alpha)$, $(\bar{T}_\alpha)$, $(\bar{T}^{\alpha\beta})$, $(\bar{T}_{\alpha\beta})$, $(\bar{T}^\beta_\alpha)$. Recall that

$$\bar{T}^{\alpha\beta} = a^{\alpha\rho} a^{\beta\lambda} \bar{T}_{\rho\lambda}, \quad \bar{T}_{\alpha\beta} = a_{\alpha\lambda} a_{\beta\rho} \bar{T}^{\lambda\rho}, \quad \bar{T}^\beta_\alpha = a^{\beta\rho} \bar{T}_{\rho\alpha}$$

Derivation of tangent basis vectors a_α or a^β and the normal vector $a^3 = a_3$ are given by Gauss and Weingarten formulas, respectively, by (Naghdi, 1970; Koiter, 1970; Do Carmo, 1976)

$$a_{\alpha,\beta} = \bar{\Gamma}^\gamma_{\alpha\beta} a_\gamma + b_{\alpha\beta} a^3; \quad a^\alpha_{,\beta} = -\bar{\Gamma}^\alpha_{\beta\gamma} a^\gamma + b^\alpha_\beta a^3; \quad a^3_{,\alpha} = a_{3,\alpha} = -b^\gamma_\alpha a_\gamma$$

where the Christoffel symbols $\bar{\Gamma}^\gamma_{\alpha\beta}$ are defined by

$$\bar{\Gamma}^\gamma_{\alpha\beta} = \bar{\Gamma}^\gamma_{\beta\alpha} = a^{\gamma\lambda} \bar{\Gamma}_{\lambda\alpha\beta} = \frac{1}{2} a^{\gamma\lambda}(a_{\lambda\beta,\alpha} + a_{\lambda\alpha,\beta} - a_{\alpha\beta,\lambda}) = a^\gamma \cdot a_{\alpha,\beta}.$$

Covariant derivations on tensors are given by

$$\nabla_\lambda \bar{T}_\alpha = \bar{T}_{\alpha,\lambda} - \bar{\Gamma}^\gamma_{\alpha\lambda} T_\gamma; \quad \nabla_\lambda \bar{T}^\alpha = \bar{T}^\alpha_{,\lambda} - \bar{\Gamma}^\alpha_{\lambda\gamma} \bar{T}^\gamma; \quad \nabla_\lambda \bar{T}^{\alpha\beta} = \bar{T}^{\alpha\beta}_{,\lambda} + \bar{\Gamma}^\alpha_{\lambda\gamma} \bar{T}^{\gamma\beta} + \bar{\Gamma}^\beta_{\lambda\gamma} \bar{T}^{\alpha\gamma};$$

$$\nabla_\lambda \bar{T}_{\alpha\beta} = \bar{T}_{\alpha\beta,\lambda} - \bar{\Gamma}^\gamma_{\alpha\lambda} \bar{T}_{\gamma\beta} - \bar{\Gamma}^\gamma_{\beta\lambda} \bar{T}_{\alpha\gamma}; \quad \nabla_\lambda \bar{T}^\alpha_\beta = \bar{T}^\alpha_{\beta,\lambda} + \bar{\Gamma}^\alpha_{\lambda\gamma} \bar{T}^\gamma_\beta - \bar{\Gamma}^\gamma_{\beta\lambda} \bar{T}^\alpha_\gamma.$$

These formulas also lead to $\nabla_\lambda a_{\alpha\beta} = \nabla_\lambda a^{\alpha\beta} = \nabla_\lambda \delta^\alpha_\beta = 0$. Let $v = v^\lambda a_\lambda + v^3 a_3 = v_\lambda a^\lambda + v_3 a^3$ be a vector field on S, then $v_{,\alpha} = (\nabla_\alpha v_\lambda - b_{\lambda\alpha} v_3) a^\lambda + (\nabla_\alpha v_3 + b^\lambda_\alpha v_\lambda) a^3 = (\nabla_\alpha v^\lambda - b^\lambda_\alpha v_3) a_\lambda + (\nabla_\alpha v^3 + b_{\lambda\alpha} v^2) a_3$ where we denote $\nabla_\alpha v^3 = v^3_{,\alpha}$ and $\nabla_\alpha v_3 = v_{3,\alpha}$. The second-order covariant deviations are commutative on v_3;

$$\nabla_{\alpha\beta}v_3 = \nabla_{\beta\alpha}v_3 = v_{3,\alpha\beta} - \bar{\Gamma}^{\lambda}_{\alpha\beta}v_{3,\lambda},$$

while they satisfy the relations

$$\nabla_{\beta\gamma}\bar{T}_\alpha - \nabla_{\gamma\beta}\bar{T}_\alpha = \bar{R}^{\lambda}_{\alpha\beta\gamma}\bar{T}_\lambda = \bar{R}_{\lambda\alpha\beta\gamma}\bar{T}^{\lambda}; \quad \bar{R}^{\lambda}_{\alpha\beta\gamma} = \bar{\Gamma}^{\lambda}_{\alpha\gamma,\beta} - \bar{\Gamma}^{\lambda}_{\alpha\beta,\gamma} + \bar{\Gamma}^{v}_{\alpha\gamma}\bar{\Gamma}^{\lambda}_{v\beta} - \bar{\Gamma}^{v}_{\alpha\beta}\bar{\Gamma}^{\lambda}_{v\gamma};$$

and

$$\bar{R}_{\lambda\alpha\beta\gamma} = a_{\lambda\rho}\bar{R}^{\rho}_{\alpha\beta\gamma} = b_{\alpha\gamma}b_{\lambda\beta} - b_{\alpha\beta}b_{\lambda\gamma}.$$

Let $\bar{u}$ be a displacement of the surface and a^*_α, a^*_3 be the covariant tangent basis vectors and normal vector, respectively, of the deformed surface, then the strain and change of curvature tensors are defined by

$$\bar{E}_{\alpha\beta} = \tfrac{1}{2}(a^*_{\alpha\beta} - a_{\alpha\beta}) \quad \text{and} \quad k_{\alpha\beta} = (b^*_{\alpha\beta} - b_{\alpha\beta}).$$

The linearized part of these tensors (see Bernadou and Ciarlet, 1976) are given by

$$e_{\alpha\beta}(\bar{u}) = \tfrac{1}{2}(\nabla_\alpha\bar{u}_\beta + \nabla_\beta\bar{u}_\alpha - 2\bar{u}_3 b_{\alpha\beta}); \tag{2.1}$$

$$k_{\alpha\beta}(\bar{u}) = \nabla_\alpha b^{v}_{\beta}\bar{u}_v + b^{v}_{\alpha}\nabla_\beta\bar{u}_v + b^{v}_{\beta}\nabla_\alpha\bar{u}_v + \nabla_\alpha\nabla_\beta\bar{u}_3 - b^{\rho}_{\alpha}b_{\rho\beta}\bar{u}_3. \tag{2.2}$$

These tensors are widely used in thin shell theory and they express the change in the first and second fundamental forms. Unlike the first two, the change in the third fundamental form has not received the same attention. In the present discussion, this tensor will be of paramount interest. Using the same linearization procedure the following results are obtained:

$$a^*_\alpha = (\delta^{\lambda}_{\alpha} + \nabla_\alpha\bar{u}^{\lambda} - b^{\lambda}_{\alpha}\bar{u}_3)a_\lambda + (\nabla_\alpha\bar{u}^3 + b^{\lambda}_{\alpha}\bar{u}_\lambda)a_3; \quad a^*_3 = -(\nabla_\alpha\bar{u}^3 + b^{\lambda}_{\alpha}\bar{u}_\lambda)a^{\alpha} + a_3$$

and by simple computation

$$Q_{\alpha\beta}(\bar{u}) = \tfrac{1}{2}(c^*_{\alpha\beta} - c_{\alpha\beta}) = \tfrac{1}{2}[b^{v}_{\alpha}\nabla_\beta b^{\rho}_{v}\bar{u}_\rho + b^{v}_{\alpha}b^{\rho}_{v}\nabla_\beta\bar{u}_\rho + b^{v}_{\beta}b^{\rho}_{v}\nabla_\alpha\bar{u}_\rho + b^{v}_{\beta}\nabla_v b^{\rho}_{\alpha}\bar{u}_\rho + b^{v}_{\alpha}\nabla_\beta\nabla_v\bar{u}_3 + b^{v}_{\beta}\nabla_\alpha\nabla_v\bar{u}_3]. \tag{2.3}$$

In cases of small displacements only, these linearized parts of the different tensors will be used.

Different formulas obtained in this section will be used in the sequel since a shell will be defined as usual, using its midsurface.

2.2. Geometry and deformation of a shell

Let S be a sufficiently smooth bounded surface as described above. Let m be the generic point of S with (x^1, x^2) as coordinates; a_1, a_2 and a_3 the covariant tangent and normal vectors at m. A shell Ω is defined by

$$\Omega = \{M \text{ in } \Omega, \quad \mathbf{OM} = \mathbf{Om} + x_3 a_3; -h < x_3 < h\}.$$

The surface S is thus the midsurface of the shell Ω. The thickness of the shell at each point m is $2h$ and may depend on $(x^1, x^2) = \bar{x}$. We shall further study the family of shells Ω^ε with $-\varepsilon h < x_3 < \varepsilon h$, where ε is a small parameter.

Let us consider a shell Ω as defined above, the covariant vectors are defined by

$$g_\alpha = \mathbf{OM}_{,\alpha} = (\delta^v_\alpha - x_3 b^v_\alpha)a_v, \quad g_3 = a_3;$$

and

$$g_1 \times g_2 = (1-2z\bar{H}+z^2\bar{K})a_1 \times a_2 = \rho(x)a_1 \times a_2, \quad x = (\bar{x}, z), \quad x^3 = z.$$

It is deduced from the definition of $\bar{H}$ and $\bar{K}$ that $\rho(x) \neq 0$ if $|z| < \min(|R_1|, |R_2|)$. In fact in this case $\rho(x) > 0$. This is true if half the thickness of the shell h satisfies $h < \min(|R_1|, |R_2|)$ at each point on the midsurface. In the sequel we shall assume that the surface S and the shell Ω are such that

$$R = (\min(|R_1|, |R_2|) \neq 0 \quad m \in \bar{S}) \quad \text{and} \quad h < R. \tag{2.4}$$

Then there exists a constant ρ_0 such that

$$\rho(x) \geqslant \rho_0 > 0 \tag{2.5}$$

and g_1, g_2, g_3 automatically constitute the covariant basis of the shell Ω. We also deduce from eqn (2.4) that $(\mu^\alpha_\beta) = (\delta^\alpha_\beta - zb^\alpha_\beta)$ is invertible and g^1, g^2, g^3 defined by $g^\alpha = (\mu^{-1})^\alpha_\lambda a^\lambda$, $g^3 = a^3$ constitute the contravariant basis of Ω. Though (μ^{-1}) can be computed exactly, for further application, we shall provide the following lemma:

Lemma 1. Let $(\mu^\alpha_\beta) = (\delta^\alpha_\beta - zb^\alpha_\beta)$, then

$$(\mu^{-1})^\alpha_\beta = \sum_{m=0}^{\infty} z^n (b^n)^\alpha_\beta, \quad (b^0)^\alpha_\beta = \delta^\alpha_\beta, \quad b^1 = b, \quad (b^n)^\alpha_\beta = b^\lambda_\beta (b^{n-1})^\alpha_\lambda = b^\alpha_\lambda (b^{n-1})^\lambda_\beta.$$

Proof. The explicit formula of (μ^{-1}) can also be written as

$$\rho(\mu^{-1})^\alpha_\beta = \delta^\alpha_\beta + z(b^\alpha_\beta - 2\bar{H}\delta^\alpha_\beta), \quad \rho(x) = (1-2z\bar{H}+z^2\bar{K}) = \det(\mu^\alpha_\beta). \tag{2.6}$$

Let us suppose there exists C_n such that

$$(\mu^{-1})^\alpha_\beta = \sum_{m=0}^{\infty} z^n (C_n)^\alpha_\beta,$$

then

$$\rho(\mu^{-1})^\alpha_\beta = \sum_{m=0}^{\infty} [z^n (C_n)^\alpha_\beta - 2\bar{H}z^{n+1}(C_n)^\alpha_\beta + z^{n+2}\bar{K}(C_n)^\alpha_\beta]$$

$$= (C_0)^\alpha_\beta + z[(C_1)^\alpha_\beta - 2\bar{H}(C_0)^\alpha_\beta] + \sum_{m=2}^{\infty} z^n[(C_n)^\alpha_\beta - 2\bar{H}(C_{n-1})^\alpha_\beta + \bar{K}(C_{n-2})^\alpha_\beta]. \tag{2.7}$$

Comparing eqns (2.6) and (2.7) leads to

$$(C_0)^\alpha_\beta = \delta^\alpha_\beta; \quad (C_1)^\alpha_\beta = b^\alpha_\beta; \quad (C_n)^\alpha_\beta = 2\bar{H}(C_{n-1})^\alpha_\beta - \bar{K}(C_{n-2})^\alpha_\beta, \quad \text{for } n \geqslant 2. \tag{2.8}$$

Let

$$(b^0)^\alpha_\beta = \delta^\alpha_\beta, \quad b^1 = b, \quad \text{and} \quad (b^n)^\alpha_\beta = b^\lambda_\beta (b^{n-1})^\alpha_\lambda = b^\alpha_\lambda (b^{n-1})^\lambda_\beta.$$

We shall prove by recurrence that

$$(C_n)^\alpha_\beta = (b^n)^\alpha_\beta.$$

Now this relation is already true for $n = 0$, $n = 1$. Assume it is verified for $n-1$, then from eqn (2.8), we have

$$(C_n)^\alpha_\beta = 2\bar{H}(b^{n-1})^\alpha_\beta - \bar{K}(b^{n-2})^\alpha_\beta, \quad \text{for } n \geqslant 2;$$
$$= 2\bar{H}b^\alpha_\lambda(b^{n-2})^\lambda_\beta - \bar{K}(b^{n-2})^\lambda_\beta\delta^\alpha_\lambda = [2\bar{H}b^\alpha_\lambda - \bar{K}\delta^\alpha_\lambda](b^{n-2})^\lambda_\beta.$$

Now $\det(b - \theta\delta^\alpha_\beta) = \theta^2 - 2\theta\bar{H} + \bar{K}$; by applying Cayley–Hamilton's theorem (Greub, 1976; Nering, 1970) we deduce that

$$b^2 - 2\bar{H} + \bar{K}I = 0 \quad \text{and} \quad \bar{K}\delta^\alpha_\lambda = 2\bar{H}b^\alpha_\lambda - b^\alpha_\rho b^\rho_\lambda.$$

Combining this with eqn (2.5) we obtain

$$(C_n)^\alpha_\beta = b^\alpha_\rho b^\rho_\lambda(b^{n-2})^\lambda_\beta = b^\alpha_\rho(b^{n-1})^\rho_\beta = b^\alpha_\rho b^\lambda_\beta(b^{n-2})^\rho_\lambda = b^\lambda_\beta(b^{n-1})^\alpha_\lambda,$$

and the results hold. ◆

We now recall all useful relations (Eisenhart, 1949; Spivak, 1975; Klingenbert, 1982).
On the metric, basis vectors and volume element:

$$g_{ij} = g_{ji} = g_i \cdot g_j; \quad g^{ij} = g^{ji} = g^i \cdot g^j; \quad g_{\alpha\beta} = \mu^\nu_\alpha \mu^\lambda_\beta \alpha_{\nu\lambda}; \quad g^{\alpha\beta} = (\mu^{-1})^\alpha_\lambda(\mu^{-1})^\beta_\rho \alpha^{\lambda\rho};$$
$$g^{\alpha 3} = g_{\alpha 3} = 0; \quad g^{33} = g_{33} = 1; \quad g_\alpha = g_{\alpha\beta}g^b, \quad g^\alpha = g^{\alpha\beta}g_\beta; \quad g^i \cdot g_j = \delta^i_j, \quad g^{ik}g_{kj} = \delta^i_j;$$
$$\mathrm{d}\Omega = \sqrt{g}\,\mathrm{d}x^1\,\mathrm{d}x^2\,\mathrm{d}z = \rho\sqrt{a}\,\mathrm{d}x^1\,\mathrm{d}x^2\,\mathrm{d}z = \rho\,\mathrm{d}S\,\mathrm{d}z; \quad g = \det(g_{ij}); \quad \mathrm{d}z = \mathrm{d}x^3;$$

On the components of tensors:

$$T^i = g^{ij}T_j, \quad T_i = g_{ij}T^j; \quad T_{ij} = g_{ik}g_{jl}T^{kl}; \quad T^{ij} = g^{ik}g^{jl}T_{kl}; \quad T^i_j = g^{ik}T_{kj};$$

On the covariant derivations and Christoffel symbols:

$$\Gamma_{ijk} = \Gamma_{jik} = g_{i,k} \cdot g_k = \tfrac{1}{2}(g_{ik,j} + g_{jk,i} - g_{ij,k}); \quad \Gamma^k_{ij} = \Gamma^k_{ji} = g^{kl}\Gamma_{ijl};$$
$$g_{i,j} = \Gamma^k_{ij}g_k, \quad g^i_{,k} = -\Gamma^i_{kj}g^j; \quad (u^l g_l)_{,i} = u^l_{/i}g_l = u_{l/i}g^l,$$
$$u^l_{/i} = u^l_{,i} + \Gamma^l_{ik}u^k \quad \text{and} \quad u_{l/i} = u_{l,i} - \Gamma^k_{li}u_k; \quad g_{ij/k} = g^{ij}_{/k} = \delta^i_{j/k} = 0;$$
$$T^{ij}_{/k} = T^{ij}_{,k} + \Gamma^i_{kl}T^{lj} + \Gamma^j_{kl}T^{il}; \quad T_{ij/k} = T_{ij,k} - \Gamma^l_{kl}T_{lj} - \Gamma^l_{kj}T_{il};$$
$$T^{ij}_{j/k} = T^i_{j,k} + \Gamma^i_{kl}T^l_j - \Gamma^l_{kj}T^i_l; \quad T_{i/jk} - T_{i/kj} = R^m_{ijk}T_m = R_{mijk}T^m;$$
$$R^m_{ijk} = \Gamma^m_{ik,j} - \Gamma^m_{ij,k} + \Gamma^p_{ik}\Gamma^m_{pj} - \Gamma^p_{ij}\Gamma^m_{pk}; \quad R_{mijk} = g_{mp}R^p_{ijk}.$$

It should be noted that the formula $T_{i/jk} - T_{i/kj}$ is the same in any curvilinear domain not having necessarily the form of Ω.

2.3. Relations between surface and three-dimensional tensors

As we mentioned in the introduction, one important aspect in our approach will be based on the different relations presented in this paragraph concerning:

2.3.1. Christoffel symbols

We have $g_{\alpha,\beta} = \mu_\alpha^\nu a_{\nu,\beta} - z b^\nu_{\alpha,\beta} a_\nu$ and $g^\gamma = (\mu^{-1})^\gamma_\nu a^\nu$, then

$$\Gamma^\gamma_{\alpha\beta} = g_{\alpha,\beta}\cdot g^\gamma = (\mu^{-1})^\gamma_\nu(\bar\Gamma^\nu_{\alpha\beta} - z(\nabla_\beta b^\nu_\alpha + \bar\Gamma^\lambda_{\alpha\beta} b^\nu_\lambda))$$

and by introducing the Taylor expansion of (μ^{-1}) (Lemma 1), we deduce the relation

$$\Gamma^\gamma_{\alpha\beta} = \bar\Gamma^\gamma_{\alpha\beta} + (\mu^{-1})^\gamma_\nu \nabla_\beta \mu^\nu_\alpha.$$

Similarly

$$\Gamma^\alpha_{\beta 3} = -(\mu^{-1})^\alpha_\lambda b^\lambda_\beta; \quad \Gamma^3_{\alpha\beta} = \mu^\nu_\alpha b_{\nu\beta}; \quad \Gamma^3_{3\beta} = \Gamma^\alpha_{33} = \Gamma^3_{33} = 0;$$

2.3.2. Vector quantities

Let $T = T_\alpha g^\alpha + T_3 g^3 = T^\alpha g_\alpha + T^3 g_3$ $(g_3 = a_3 = a^3)$; we can also write

$$T = \bar T_\alpha a^\alpha + \bar T_3 = \bar T^\alpha a_\alpha + \bar T^3 a_3,$$

since a_1, a_2, a_3 and a^1, a^2, a^3 constitute basis in IR^3. From this we obtain the following relations:

$T^3 = T_3 = \bar T^3 = \bar T_3$; $T_\alpha = \mu^\nu_\alpha \bar T_\nu$, $T^\alpha = (\mu^{-1})^\alpha_\nu \bar T^\nu$, $\bar T_\alpha = (\mu^{-1})^\nu_\alpha T_\nu$, $\bar T^\alpha = \mu^\alpha_\nu T^\nu$. In like manner we obtain

$$T^\alpha_\beta = (\mu^{-1})^\alpha_\nu \mu^\lambda_\beta \bar T^\nu_\lambda, \quad \bar T^\alpha_\beta = \mu^\alpha_\nu (\mu^{-1})^\lambda_\beta T^\nu_\lambda, \quad T^\alpha_3 = (\mu^{-1})^\alpha_\nu \bar T^\nu_3, \quad \bar T^\alpha_3 = \mu^\alpha_\nu T^\nu_3,$$
$$T^3_\alpha = \mu^\nu_\alpha \bar T^3_\nu, \quad \bar T^3_\alpha = (\mu^{-1})^\nu_\alpha T^3_\nu, \quad T^3_3 = \bar T^3_3.$$

2.3.3. Derivations

$$T_{\alpha/\beta} = T_{\alpha,\beta} - \Gamma^\lambda_{\alpha\beta} T_\lambda - \Gamma^3_{\alpha\beta} T_3 = \mu^\nu_\alpha[\nabla_\beta \bar T_\nu - b_{\nu\beta}\bar T^3],$$
$$T^\alpha_{/\beta} = T^\alpha_{,\beta} + \Gamma^\alpha_{\beta\lambda} T^\lambda + \Gamma^\alpha_{\beta 3} T^3 = (\mu^{-1})^\nu_\alpha[\nabla_\beta \bar T^\nu - b^\nu_\beta \bar T^3];$$
$$T_{\alpha/3} = \mu^\nu_\alpha \bar T_{\nu,3}, \quad T_{3/\alpha} = \bar T_{3,\alpha} + b^\lambda_\alpha \bar T_\lambda; \quad T^\alpha_{/3} = (\mu^{-1})^\alpha_\nu \bar T^\nu_{,3}, \quad T^3_{/\alpha} = \bar T^3_{,\alpha} + b_{\alpha\lambda}\bar T^\lambda;$$
$$T^3_{/3} = T_{3/3} = T_{3,3} = \bar T^3_{,3} = \bar T_{3,3}. \tag{2.9}$$

2.3.4. Linearized strain tensor

Using eqn (2.9) we obtain:

$$\epsilon_{\alpha\beta}(u) = \tfrac{1}{2}(u_{\alpha/\beta} + u_{\beta/\alpha}) = \tfrac{1}{2}[\mu^\nu_\alpha(\nabla_\beta \bar u_\nu - b_{\nu\beta}\bar u_3) + \mu^\nu_\beta(\nabla_\alpha \bar u_\nu - b_{\nu\alpha}\bar u_3)],$$
$$\epsilon_{\alpha 3}(u) = \tfrac{1}{2}(u_{\alpha/3} + u_{3/\alpha}) = \tfrac{1}{2}[\mu^\nu_\alpha \bar u_{\nu,3} + (\bar u_{3,\alpha} + b^\nu_\alpha \bar u_\nu)],$$
$$\epsilon_{33}(u) = u_{3,3} = \bar u_{3,3}. \tag{2.10}$$

It should be noted here that all quantities expressed with $^-$ also depend on $z = x_3$. We have crucially used the expansion of (μ^{-1}) and the relations on Christoffel symbols to obtain eqn (2.10) which shall be of great importance in Section 3. In most recent works only truncated parts of these formulas have been expressed (Ciarlet and Lods, 1996).

2.4. Functional spaces

Let S and Ω be sufficiently regular as described above. We shall denote the border of S by $\gamma = \partial S$ and assume $\bar{\gamma} = \bar{\gamma}_0 \bigcup \bar{\gamma}_1$ where γ_0 and γ_1 are parts of γ with a non-zero measure; $\bar{\gamma}$, $\bar{\gamma}_0$, and $\bar{\gamma}_1$ denote their closure. We shall also denote by Γ_0 a non-zero measure subset of the border of Ω. All covariant derivation on the surface S and the domain Ω will be denoted by ∇_α and $/_j$, respectively. Derivatives should be understood in the sense of distribution (Adams, 1975). Covariant derivations on a tensor may be taken on either the covariant, contravariant or mixed components. In any case the results are indifferently the same.

To begin with, we recall that the different geometric tensors $(a_{\alpha\beta})$, $\bar{\Gamma}^{\gamma}_{\alpha\beta}$, (μ^{α}_{β}), $(\mu^{-1})^{v}_{\alpha}$, $(b_{\alpha\beta})$ together with their covariant or usual derivatives are uniformly bounded because of the regularity of the chart. Consequently the metrics and Christoffel symbols of the shell are also bounded in the same way. Next we define the spaces

$$
\begin{aligned}
H^1(S) &= \{\eta \text{ in } L^2(S), \nabla_\alpha \eta \text{ in } L^2(S)\}; \quad IH^1(S) = [H^1(S)]^2; \\
IH^1_{\gamma_0}(S) &= \{(\eta_\alpha) \text{ in } IH^1(S), \eta_\alpha = 0 \text{ on } \gamma_0\}; \\
H^1(\Omega) &= \{v^i \text{ in } L^2(\Omega), v^i|_j \text{ in } L^2(\Omega)\}; \quad IH^1(\Omega) = [H^1(\Omega)]^3; \\
IH^1_{\Gamma_0}(\Omega) &= \{v \text{ in } IH^1(\Omega), v = 0 \text{ on } \Gamma_0\}; \\
H^2(S) &= \{\eta \text{ in } H^1(S), \nabla_\alpha \eta \text{ in } H^1(S)\}; \\
H^2_{\gamma_0}(S) &= \{\eta \text{ in } H^2(S), \eta = \partial_{\mathbf{v}} \eta = 0 \text{ on } \gamma_0\};
\end{aligned} \tag{2.11}
$$

$\mathbf{v}$ is the unit outer normal vector on the border. The norms on $IH^1(S) \times H^2(S)$ and $IH^1(\Omega)$ are defined either by the covariant or usual derivatives by (Bernadou and Ciarlet, 1976).

$$
\|(\eta_\alpha, \eta_3)\|^2 = \sum_{i=1}^{3} \|\eta_i\|^2 L^2(S) + \sum_{\alpha,\beta} \|\nabla_\alpha \eta_\beta\|^2 L^2(S) + \sum_{\alpha} \|\nabla_\alpha \eta_3\|^2 L^2(S) + \sum_{\alpha,\beta} \|\nabla_{\alpha\beta} \eta_3\|^2 L^2(S);
$$

$$
\|v\|^2_{1,\Omega} = \sum_{i,j=1}^{3} \|v_i\|^2 L^2(\Omega) + \|v_{i/j}\|^2 L^2(\Omega).
$$

We recall that if (T^{ij}) is a tensor then the L^2-norm is given by (Rougee, 1969),

$$
\|T\|^2 = \int_\Omega T^{ij} T_{ij} \, d\Omega = \int_\Omega T^i_j T^j_i \, d\Omega.
$$

We also recall the following lemma found in Bernadou and Ciarlet (1976).

Lemma 2. Let $\eta = (\eta_\alpha, \eta_3)$ and $e(\eta)$, $k(\eta)$ be defined through the formulas (2.1) and (2.2), then the semi-norm $|\,.\,|$ defined by

$$
|(\eta_\alpha, \eta_3)|^2 = \left(\sum_{\alpha,\beta} \|e_{\alpha\beta}(\eta)\|^2 L^2(S) + \sum_{\alpha,\beta} \|k_{\alpha\beta}(\eta)\|^2 L^2(S) \right) \tag{2.12}
$$

is equivalent to the induced $IH^1(S) \times H^2(S)$-norm, in $IH^1_{\gamma_0}(S) \times H^2_{\gamma_0}(S)$. ◆

Lemma 3 (*rigid body displacement*). Let Ω be a sufficiently smooth bounded domain in IR^3 and let v be in $IH^1(\Omega)$, then the following two propositions are equivalent:

(i) $v(x) = A + B \times \mathbf{OM}$, x in $\bar{\Omega}$, A and B constant vectors in IR^3;
(ii) $\epsilon_{ij}(v) = \frac{1}{2}(u_{i/j} + u_{j/i}) = 0$.

Proof. We first recall that $g_i \times g_j = \varepsilon_{ijk} g^k$ and $g^i \times g^j = \varepsilon^{ijk} g_k$ where $\varepsilon^{ijk} = 1/\sqrt{g} e^{ijk}$, $\varepsilon_{ijk} = \sqrt{g} e_{ijk}$, $g = \det(g_{ij})$; e^{ijk} and e_{ijk} are the permutation symbols.

Now if $v(x) = A + B \times \mathbf{OM}$, then $v_{,i} = B \times \mathbf{OM}_{,i} = B \times g_i$. Let $B = B^m g_m$, then $v_{,i} = v_{j/i} g^j = B^m g_m \times g_i = B^m \varepsilon_{mij} g^j$.

Therefore $v_{j/i} = B^m \varepsilon_{mij}$ and $(v_{i/j} + v_{j/i})/2 = B^m(\varepsilon_{mij} + \varepsilon_{mji})/2 = \epsilon_{ij}(v) = 0$.

So (i) $\Rightarrow$(ii) is thus proved. Suppose $\epsilon_{ij}(v) = 0$. We shall first show that the vector $B = B^m g_m$, where $B^m = \varepsilon^{mij}\Omega_{ij}(v)/2$, and $\Omega_{ij}(v) = (v_{i/j} - v_{j/i})/2$ is constant if $\epsilon_{ij}(v) = 0$. Since $\varepsilon^{mij}_{/k} = 0$ we deduce that $B_{,k} = B^m_{/k} g_m = 1/2 \varepsilon^{mij} \Omega_{ij/k} g_m$.

But by computation $\Omega_{ij/k}(v) - \epsilon_{ik/j}(v) + \epsilon_{jk/i}(v) = R^m_{ijk} v_m = g_i \cdot (v_{,kj} - v_{,jk})$. This is the curvilinear version of the well-known formula in Cartesian coordinate system (Germain and Müller, 1980). If v in $IH^1(\Omega)$, then in the sense of distribution $v_{,kj} - v_{,jk} = 0$ (Lions and Magenes, 1968). We therefore deduce that

$$\Omega_{ij/k}(v) = \epsilon_{ik/j}(v) - \epsilon_{jk/i}(v) = 0$$

and the vector B is constant. Let B be defined as above, then

$$(B \times \mathbf{OM})_{,k} = B \times g_k = 1/2 \varepsilon^{mji} \Omega_{ij}(v) g_m \times g_k = 1/2 \varepsilon^{mji} \varepsilon_{mkl} \Omega_{ij}(v) g^1 = \Omega_{lk}(v) g^1 = v_{l/k} g^1,$$

since $\epsilon_{lk}(v) = 0$. So $(v - B \times \mathbf{OM})_{,k} = 0$ and this implies that there exists a constant vector A such that $v(x) = A + B \times \mathbf{OM}$, so (ii) $\Rightarrow$(i) is verified and the proof is completed. ◆

Lemma 4 (*Korn's inequality*). The semi-norm $|v|$ defined on $IH^1_{\Gamma_0}(\Omega)$ by

$$|v|^2 = \|\epsilon(v)\|^2 L^2_s(\Omega) = \int_\Omega \epsilon^{ij}(v) \epsilon_{ij}(v) \, d\Omega \tag{2.13}$$

is equivalent to the IH^1-normal $\|v\|_{1,\Omega}$.

Proof. If $|v| = 0$, then from Lemma 3, $v(x) = A + B \times \mathbf{OM}$. If $v = 0$ on Γ_0 then it is easy to check that $v = 0$. So $|\,.\,|$ defines a semi-norm in $IH^1_{\Gamma_0}(\Omega)$. Clearly there exists a constant C such that

$$|v| \leqslant C \|v\|_{1,\Omega}.$$

We shall show that there exists a constant C such that

$$\|v\|_{1,\Omega} < C|v|.$$

Assume that this last inequality is false. Then these exists a sequence v^n such that

$$\|v^n\|_{1,\Omega} = 1 \quad \text{and} \quad |v^n| \leqslant 1/n, \quad n \in IN.$$

We can extract a subsequence still denoted v^n for simplicity such that $v^n \overset{w}{\to} v$ ($\overset{w}{\to}$denotes weak convergence). We deduce from the semi-continuity of $\|\,.\,\|_{1,\Omega}$ and $|\,.\,|$ that

$$\|v\|_{1,\Omega} = 1 \quad \text{and} \quad |v| = 0.$$

This is contradictory, since from Lemma 3, $|v| = 0$ implies that $v = 0$ and the lemma is proved. ◆

This type of argument is frequent in recent works. We shall frequently use Lax Milgram's theorem in establishing existence and uniqueness of coercitive linear variational equations, i.e., variational problems of the form $a(u, v) = L(v)$ in which $a(.,.)$ is a continuous bilinear and coercitive form while $L(.)$ is a linear continuous form. Detail information on distributions, functional spaces and their applications used here can be found in Brezis (1983), Schwartz (1966), Adams (1975) and others (see references). Very many versions of Korn's inequality on a surface now exist in recent works. Some of the constants depend on the thickness small parameter ε^{-1}. It should be remarked that no such estimations occur here. Korn's inequality in Ω can also be proved but in a much more laborious way by using the Cartesian version.

3. The three-dimensional problem

3.1. The unscaled problem posed over Ω^ε

In our subsequent analysis, we consider a shell as defined in Section 1 (Fig. 1). For simplicity we assume $h = 1m$ and let

$$\Omega^\varepsilon = S \times]-\varepsilon, \varepsilon[, \partial\bar{\Omega}^\varepsilon = \bar{\Gamma}_0^\varepsilon \bigcup \bar{\Gamma}_-^\varepsilon \bigcup \bar{\Gamma}_+^\varepsilon \bigcup \bar{\gamma}_1 \times [-\varepsilon, \varepsilon];$$

$$\Gamma_0^\varepsilon = \gamma_0 \times]-\varepsilon, \varepsilon[, \Gamma_1^\varepsilon = \gamma_1 \times]-\varepsilon, \varepsilon[\bigcup \bar{\Gamma}_-^\varepsilon \bigcup \bar{\Gamma}_+^\varepsilon; \quad \bar{\Gamma}_-^\varepsilon = \bar{S} \times \{-\varepsilon\}; \quad \bar{\Gamma}_+^\varepsilon = \bar{S} \times \{\varepsilon\}$$

be a sufficiently smooth bounded open subset of IR^3, as described in Section 1, with ε being a small parameter. We recall that ε is dimensionless because we have chosen $h = 1m$. We shall use

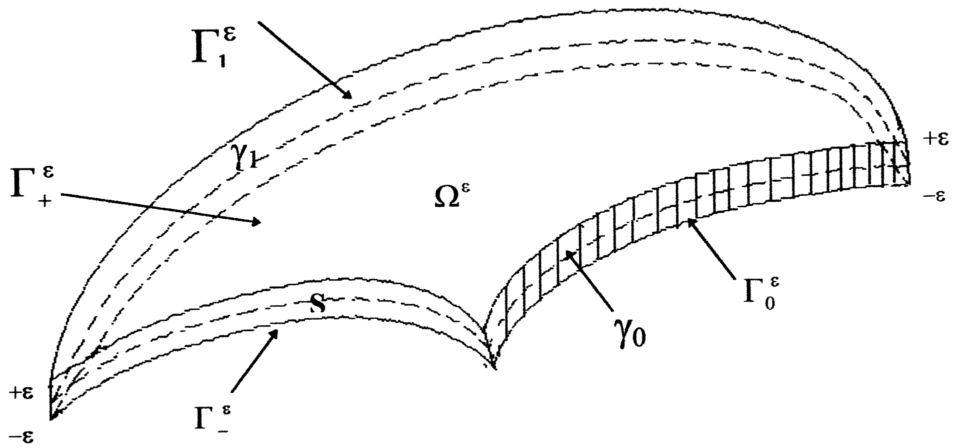

Fig. 1. The three-dimensional clamped shell. The set $\bar{\Omega}^\varepsilon = S \times [-\varepsilon, +\varepsilon]$ is the reference configuration of a shell, with thickness 2ε and midsurface S, clamped on the portion $\Gamma_0^\varepsilon = \gamma_0 \times [-\varepsilon, +\varepsilon]$ of its lateral surface $\gamma \times [-\varepsilon, +\varepsilon]$, ($\gamma = \partial S$ is the border of S). Body forces (f_i^ε) are applied in the shell's interior $\Omega^\varepsilon = S \times]-\varepsilon, +\varepsilon[$. Surface forces ($g_i^\varepsilon$) are applied on $\Gamma_1^\varepsilon = \gamma_1 \times [-\varepsilon, +\varepsilon] \bigcup \Gamma^\varepsilon \bigcup \Gamma_+^\varepsilon$. $\Gamma_-^\varepsilon = S \times \{-\varepsilon\}$ and $\Gamma_+^\varepsilon = S \times \{+\varepsilon\}$ are, respectively, the lower and upper faces while $\gamma_1 \times]-\varepsilon, +\varepsilon[$ is the free part of the lateral surface.

the traction–displacement problem as our model-problem. The three-dimensional equilibrium equations of an elastic isotropic homogeneous shell which occupies a domain Ω^ε are:

$$\begin{aligned}
&-\operatorname{div}\sigma^\varepsilon = f^\varepsilon \quad \text{in } \Omega^\varepsilon, \quad u^\varepsilon = 0 \quad \text{on } \Gamma_0^\varepsilon, \quad \sigma^\varepsilon \mathbf{n}^\varepsilon = g^\varepsilon, \quad \text{on } \bar{\Gamma}_1^\varepsilon;\\
&\qquad g^\varepsilon = \bar{h}_-^\varepsilon \quad \text{on } \bar{\Gamma}_-^\varepsilon; \quad g^\varepsilon = \bar{h}_+^\varepsilon \quad \text{on } \bar{\Gamma}_+^\varepsilon; \quad g^\varepsilon = \bar{h}^\varepsilon \quad \text{on } \bar{\gamma}_1 \times [-\varepsilon, \varepsilon],\\
&\epsilon_{ij}(u^\varepsilon) = (u^\varepsilon_{i|j} + u^\varepsilon_{j|i})/2,\\
&A_\varepsilon^{ijkl}(x^\varepsilon) = \Lambda^\varepsilon g_\varepsilon^{ij} g_\varepsilon^{kl} + G^\varepsilon (g_\varepsilon^{ik} g_\varepsilon^{jl} + g_\varepsilon^{il} g_\varepsilon^{jk}); \quad \Lambda^\varepsilon, \quad G^\varepsilon > 0,\\
&\sigma^{ij}(x^\varepsilon) = A_\varepsilon^{ijkl}(x^\varepsilon)\epsilon_{kl}(u^\varepsilon), \quad \sigma^\varepsilon = (\sigma^{ij}(x^\varepsilon)).
\end{aligned} \tag{3.1}$$

In these equations, f^ε and g^ε are, respectively, the volume force and surface force, n^ε the unit outer normal vector, Λ^ε, G^ε the Lamé moduli, (ϵ_{ij}) the linearized deformation tensor, g_ε^{ij} the metric tensor on Ω^ε and $A_\varepsilon^{ijkl}(x^\varepsilon)$ is the modulus tensor.

Let $IH^1_{\Gamma_0^\varepsilon} = IH^1_{\Gamma_0^\varepsilon}(\Omega^\varepsilon)$ be defined as in eqn (2.11) on Ω^ε, the equilibrium eqn (3.1) is also equivalent to the variational equation:

$$\begin{aligned}
&u^\varepsilon \quad \text{in } IH^1_{\Gamma_0^\varepsilon}\\
&\int_{\Omega_\varepsilon} \sigma^\varepsilon(u^\varepsilon) : \epsilon(v)\, \mathrm{d}\Omega^\varepsilon = \int_{\Omega^\varepsilon} f^\varepsilon \cdot v \, \mathrm{d}\Omega^\varepsilon + \int_{\Gamma_1^\varepsilon} g^\varepsilon \cdot v \, \mathrm{d}s = L^\varepsilon(v), \forall v \quad \text{in } IH^1_{\Gamma_0^\varepsilon}
\end{aligned} \tag{3.2}$$

where

$$A : B = A^{ij} B_{ij} = A_{ij} B^{ij} = A^i_j B^j_1; \quad f \cdot v = f^i v_i = f_i v^i.$$

Theorem 1. We assume f_i^ε is in $L^2(\Omega^\varepsilon)$, g_i^ε in $L^2(\Gamma_1^\varepsilon)$, then the variational eqn (3.2) has a unique solution.

Proof. The modulus tensor defined in eqn (3.1) is symmetric positive definite and there exists a constant $C(\varepsilon)$ such that

$$L^\varepsilon(u^\varepsilon) = \int_{\Omega^\varepsilon} \sigma^\varepsilon(u^\varepsilon) : \epsilon(u^\varepsilon)\, \mathrm{d}\Omega^\varepsilon = a(\varepsilon)(u^\varepsilon, u^\varepsilon) \geqslant C(\varepsilon) \int_{\Omega^\varepsilon} \epsilon(u^\varepsilon) : \epsilon(u^\varepsilon)\, \mathrm{d}\Omega^\varepsilon = C(\varepsilon)|u^\varepsilon|^2,$$

where L^ε is a continuous linear from and $a(\varepsilon)(.,.)$ is a continuous bilinear form. We deduce from Korn's inequality (Lemma 4) that the bilinear form is coercive. The result is a classical application of Lax Milgram's theorem. ◆

3.2. *The scaled problem posed over* Ω

We consider the mapping (Fig. 2)

$$\phi^\varepsilon : x \quad \text{in } \bar{\Omega} = \bar{S} \times [-1, 1] \to x^\varepsilon = (x_1, x_2, \varepsilon x_3) \quad \text{in } \bar{\Omega}^\varepsilon = \bar{S} \times [-\varepsilon, +\varepsilon]$$

where x and x^ε are coordinate systems in the closed sets $\bar{\Omega}$ and $\bar{\Omega}^\varepsilon$ respectively. If v^ε is a vector field defined in $\bar{\Omega}^\varepsilon$, then $v^\varepsilon \circ \phi^\varepsilon$ defines a vector field in $\bar{\Omega}$. All variables X^ε or X_ε are related to $\bar{\Omega}^\varepsilon$, while X or $X(\varepsilon)$ are related to $\bar{\Omega}$. We recall that all Greek indices range in $\{1, 2\}$ while Latin indices range in $\{1, 2, 3\}$.

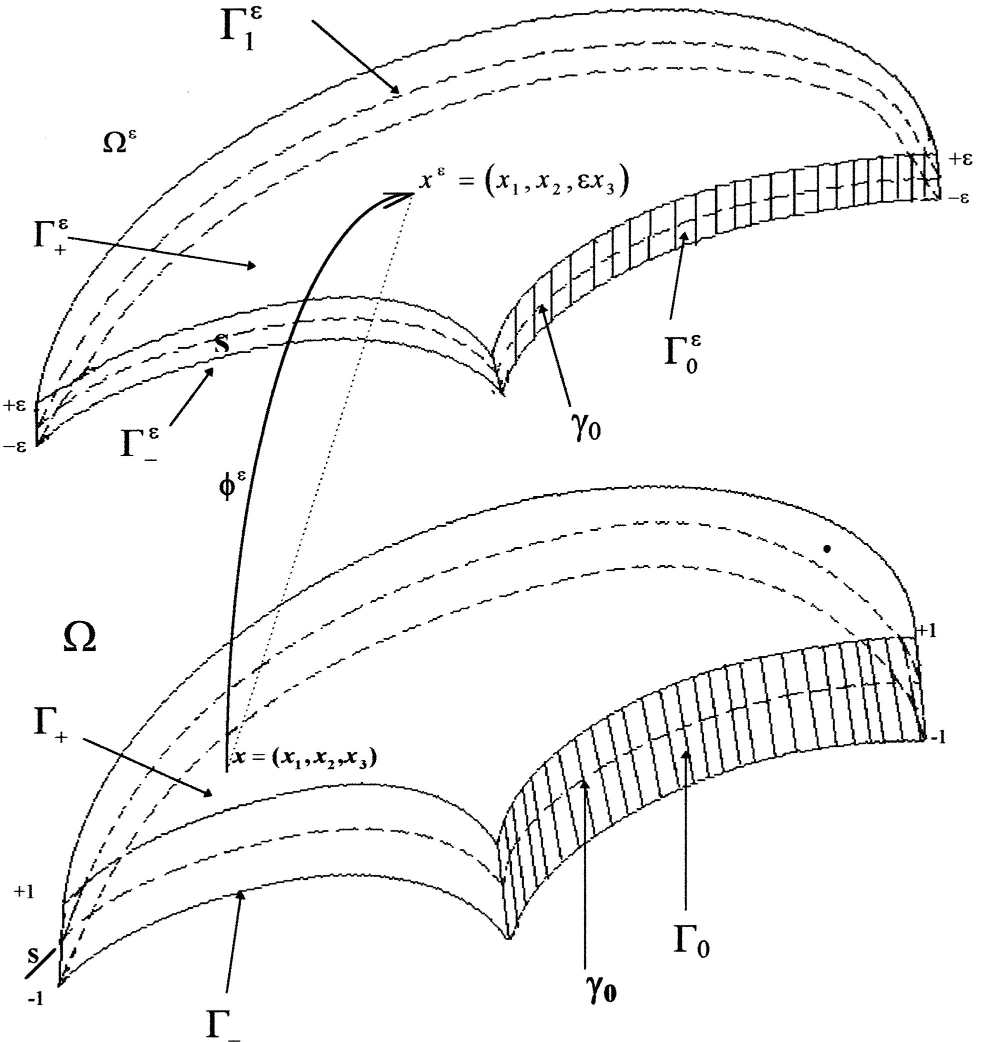

Fig. 2. The three-dimensional unscaled and scaled clamped shell. Each point $x^\varepsilon = (x_1, x_2, x_3^\varepsilon)$ of the reference configuration $\bar{\Omega}^\varepsilon = \bar{S} \times [-\varepsilon, +\varepsilon]$ is the image $\phi^\varepsilon(x)$ of the point $x = (x_1, x_2, \varepsilon^{-1} x_3^\varepsilon)$ of the set $\bar{\Omega} = \bar{S} \times [-1, +1]$. The set $\bar{\Omega}$ is independent of ε. $\Omega^3 = S \times]-\varepsilon, +\varepsilon[$, $\Gamma_0^\varepsilon = \gamma_0 \times [-\varepsilon, +\varepsilon]$; $\Omega = S \times]-1, +1[$, $\Gamma_0 = \gamma_0 \times [-1, +1]$, $\Gamma_+^\varepsilon = S \times \{+\varepsilon\}$, $\Gamma_-^\varepsilon = S \times \{-\varepsilon\}$; $\Gamma_+ = \mathrm{S} \times \{+1\}$, $\Gamma_- = \mathrm{S} \times \{-1\}$, $\Gamma_-^\varepsilon = \gamma_1 \times [-\varepsilon, +\varepsilon] \bigcup \Gamma_-^\varepsilon \bigcup \Gamma_+^\varepsilon$; $\gamma = \partial S$; $\Gamma_- = \gamma_1 \times [-1, +1] \bigcup \Gamma_- \bigcup \Gamma_+$. Γ_- is the lower face of the scaled shell, Γ_+ is the upper face of the scaled shell. $\gamma_1 \times [-1, +1]$ is the part of its lateral surface.

We define the scaling as follows:

on the displacement

$$u_\alpha^\varepsilon(x^\varepsilon) = \varepsilon^2 u_\alpha(\varepsilon)(x), \quad u_3^\varepsilon(x^3) = \varepsilon u_3(\varepsilon)(x), \quad \text{for all } x^\varepsilon = \phi^\varepsilon x \in \bar{\Omega}^\varepsilon; \tag{3.3}$$

on the forces

$$f^{\varepsilon}_{\alpha}(x^{\varepsilon}) = \varepsilon^{-1} f_{\alpha}(x), \quad f^{\varepsilon}_{3}(x^{\varepsilon}) = f_3(x), \quad \text{for all } x^{\varepsilon} = \phi^{\varepsilon} x \in \Omega^{\varepsilon},$$

$$g^{\varepsilon}_{\alpha}(x^{\varepsilon}) = g_{\alpha}(x), \quad g^{\varepsilon}_{3}(x^{\varepsilon}) = \epsilon g_3(x), \quad \text{for all } x^{\varepsilon} = \phi^{\varepsilon} x \in \Gamma^{\varepsilon}_{1}; \tag{3.4}$$

on the geometry of the surface (on curvature tensor)

$$b^{\varepsilon}_{\alpha\beta} = \varepsilon b_{\alpha\beta}, \quad b^{\varepsilon\alpha}_{\beta} = \frac{1}{\varepsilon} b^{\alpha}_{\beta}; \tag{3.5}$$

on the Lamé constants

$$\Lambda^{\varepsilon} = \varepsilon^{-3} \Lambda, \quad G^{\varepsilon} = \varepsilon^{-3} G; \tag{3.6}$$

or on the Young's modulus and Poisson's coefficient

$$E^{\varepsilon} = \varepsilon^{-3} E, \quad \bar{v}^{\varepsilon} = \bar{v}. \tag{3.7}$$

As a consequence of the above scalings we obtain the following relations:

$$(\mu_{\varepsilon})^{\alpha}_{\beta} = (\delta^{\alpha}_{\beta} - z^{\varepsilon} b^{\varepsilon\alpha}_{\beta}) = (\delta^{\alpha}_{\beta} - z b^{\alpha}_{\beta}) = \mu^{\alpha}_{\beta}; \quad (\mu^{-1}_{\varepsilon})^{\alpha}_{\beta} = (\delta^{\alpha}_{\beta} - z b^{\alpha}_{\beta})^{-1} = (\mu^{-1})^{\alpha}_{\beta};$$

$$g^{\alpha\beta}_{\varepsilon}(x^{\varepsilon}) = g^{\alpha\beta}(x), \quad g^{\alpha 3}_{\varepsilon}(x^{\varepsilon}) = g^{\alpha 3}(x) = 0, \quad g^{33}_{\varepsilon}(x^{\varepsilon}) = g^{33}(x) = 1;$$

$$\epsilon_{\alpha\beta}(u^{\varepsilon}(x^{\varepsilon}) = \tfrac{1}{2}[(\mu_{\varepsilon})^{v}_{\alpha}(\nabla_{\beta}\bar{u}^{\varepsilon}_{v} - b^{\varepsilon}_{v\beta}\bar{u}^{\varepsilon}_{3}) + (\mu_{\varepsilon})^{v}_{\alpha}(\nabla_{\alpha}\bar{u}^{\varepsilon}_{v} - b^{\varepsilon}_{v\alpha}\bar{u}^{\varepsilon}_{3})],$$

$$= \varepsilon^{2}\tfrac{1}{2}[\mu^{v}_{\alpha}(\nabla_{\beta}\bar{u}_{v} - b_{v\beta}\bar{u}_{3}) + \mu^{v}_{\beta}(\nabla_{\alpha}\bar{u}_{v} - b_{v\alpha}\bar{u}_{3})] = \varepsilon^{2}\epsilon_{\alpha\beta}(u(\varepsilon)(x)),$$

$$\epsilon_{\alpha 3}(u^{\varepsilon}(x^{\varepsilon})) = \tfrac{1}{2}(u^{\varepsilon}_{\alpha/3} + u^{\varepsilon}_{3/\alpha}) = \varepsilon\tfrac{1}{2}[\mu^{v}_{\alpha}\bar{u}_{v,3} + (\bar{u}_{3,\alpha} + b^{v}_{\alpha}\bar{u}_{v})] = \varepsilon\epsilon_{\alpha 3}(u(\varepsilon)(x)),$$

$$\epsilon_{33}(u^{\varepsilon}(x^{\varepsilon})) = u_{3,3} = \bar{u}_{3,3} = \epsilon_{33}(u(\varepsilon)(x)); \quad d\Omega^{\varepsilon} = \varepsilon\rho(x)\, dS\, dz = \varepsilon\, d\Omega. \tag{3.8}$$

We also denote by $\bar{\Gamma}_0$, $\bar{\Gamma}_1 = \bar{\Gamma}_- \bigcup \bar{\Gamma}_+ \bigcup \bar{\gamma}_1 \times [-1, 1]$ the corresponding subsets of the scaled border $\partial\bar{\Omega}$. As a consequence of eqns (3.5) and (3.6) the modulus tensor now satisfies

$$A^{ijkl}_{\varepsilon}(x^{\varepsilon}) = \varepsilon^{-3} A^{ijkl}(x),$$

where $A^{ijkl}(x)$ is nothing but $A^{ijkl}_{1}(x)$ with Λ^{ε} and G^{ε} replaced by Λ and G. It is also symmetric positive definite and elliptic i.e. there exists a constant $C > 0$ such that

$$A^{ijkl}(x)\epsilon_{ij}\epsilon_{kl} > C\epsilon_{ij}\epsilon^{ij}. \tag{3.9}$$

The hypothesis (3.5) also implies that on the scaled domain Ω, the covariant and contravariant basis vector are g^{i} and g_{i} unlike the case in Ciarlet and Miara (1992) where they were replaced by $g_{i,\varepsilon}$ and $g^{i,\varepsilon}$. In their analysis only a truncated part of the Taylor expansion of $g^{i,\varepsilon}$ appeared in the limit analysis. The Shallow shell equations are thus justified. The scalings found in eqns (3.3) and (3.4) and (3.6) and (3.7) have been widely used and justified at length by Ciarlet et al. (1989), Ciarlet (1990) and others (see references). In fact they appear to be the appropriate scalings which naturally lead to the justification of Kirchhoff–Love models in linear and non-linear elasticity for thin shells and plates. Unlike the others, the scaling found in eqn (3.5) has not been widely used. Naghdi (1970), Green and Zerna (1968), Destuynder (1986) used this type of scaling but in a different framework. In fact this scaling is unusual but appears to be the best way to consider the fact that in a thick shell, the scaled domain is locally a three-dimensional body with lesser or no

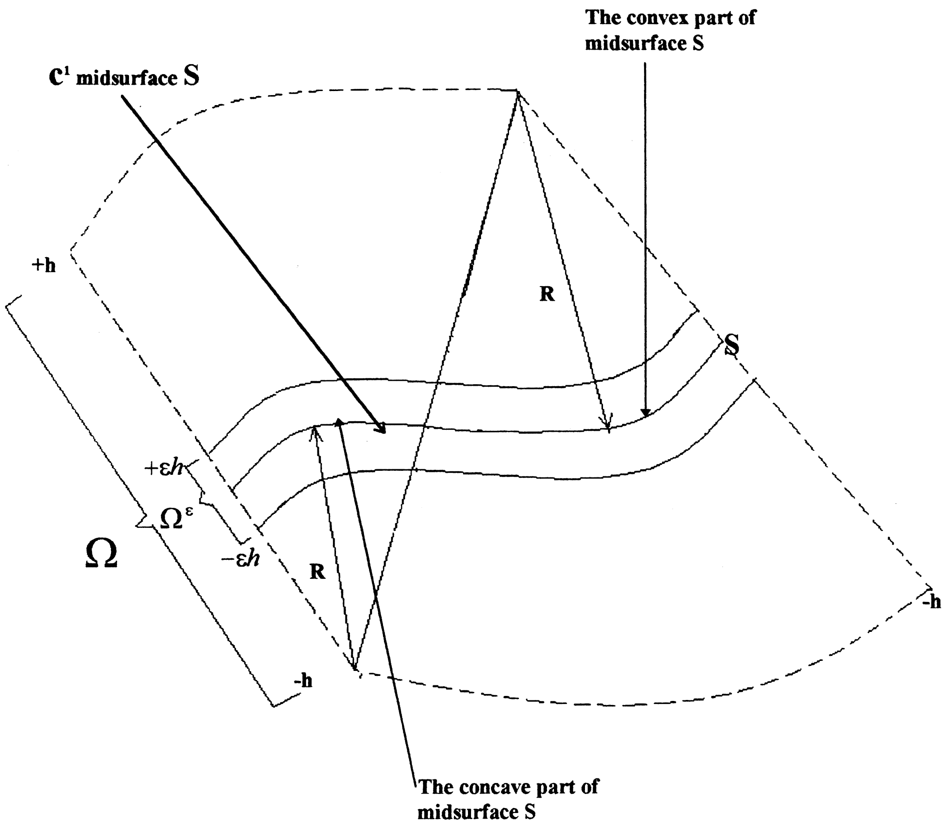

Fig. 3. Cross section of locally unscaled and scaled slab. N.B. A locally thick slab with (at least) C^1 midsurface S made of two parts: one convex and the other concave becomes almost parallelipipedic when scaled.

curvature. To demonstrate, consider an initial domain Ω^ε whose midsurface S^ε has the diagonal curvature tensor (b^α_β); $b^1_1 = 1/r = b^2_2$. Then this same surface in the scaled domain has the diagonal curvature tensor (B^α_β); $B^1_1 = 1/R = B^2_2$. The scaling implies that $R = r/\varepsilon$, that is, the curvature becomes small when the domain is scaled. For example, locally, a thick spherical slab with a C^1 midsurface made of two parts: one convex and the other concave; when scaled, is almost parallelipedic (Fig. 3).

Using the scalings defined above the scale problem is now equivalent to find $u(\varepsilon)$ in $IH^1_{\Gamma_0}$;

$$\int_\Omega [\Lambda\epsilon^\tau_\tau(u(\varepsilon))\epsilon^\beta_\beta(v) + 2G\epsilon_{\alpha\beta}(u(\varepsilon))\epsilon^{\alpha\beta}(v)]\,\mathrm{d}\Omega$$

$$\frac{1}{\varepsilon^2}\int_\Omega [\Lambda\epsilon^\tau_\tau(u(\varepsilon))\epsilon^3_3(v) + \Lambda\epsilon^3_3(u(\varepsilon))\epsilon^\beta_\beta(v) + 4G\epsilon_{\alpha 3}(u(\varepsilon))\epsilon^{\alpha 3}(v)]\,\mathrm{d}\Omega$$

$$\frac{1}{\varepsilon^4}\int_\Omega [(\Lambda + 2G)\epsilon_{33}(u(\varepsilon))\epsilon^{33}(v) +]\,\mathrm{d}\Omega = \int_\Omega f^i v_i\,\mathrm{d}\Omega + \int_{\Gamma_1} g^i v_i\,\mathrm{d}s = L(v), \quad v\epsilon IH^1_{\Gamma_0}. \tag{3.10}$$

Theorem 2. The variational equation (3.10) has a unique solution in $IH^1_{\Gamma_0}$.

Proof. From the scalings, f_i is in $L^2(\Omega)$, g_i in $L^2(\Gamma_1) \subseteq H^{-1/2}(\Gamma_1)$ and there exist constants C and ε_0 such that for $\varepsilon < \varepsilon_0$

$$L(u(\varepsilon)) = a(\varepsilon)(u(\varepsilon), \quad u(\varepsilon)) \geqslant C\int_{\Omega} \epsilon(u(\varepsilon)) : \epsilon(u(\varepsilon))\,\mathrm{d}\Omega = C|u(\varepsilon)|^2;$$

again using Korn's inequality (Lemma 4) and by applying Lax Milgram's theorem, we deduce the existence and uniqueness of $u(\varepsilon)$. ◆

The solution $u(\varepsilon)$ of the variational problem satisfies formally the equations:

$$\sigma^{ij}_{/j}(\varepsilon) + f^i = 0 \quad \text{in } \Omega, \tag{3.11}$$

$$u_i(\varepsilon) = 0 \quad \text{on } \Gamma_0, \tag{3.12}$$

$$\sigma(\varepsilon)\mathbf{n} = g \quad \text{on } \Gamma_1,$$

$$K^{\alpha\beta}(\varepsilon) = \epsilon^{\alpha\beta}(u(\varepsilon)), \quad K^{\alpha}_{\beta}(\varepsilon) = \epsilon^{\alpha}_{\beta}(u(\varepsilon)), \tag{3.13}$$

$$K^{\alpha 3}(\varepsilon) = \varepsilon^{-1}\epsilon^{\alpha 3}(u(\varepsilon)), \quad K^{\alpha}_{3}(\varepsilon) = \varepsilon^{-1}\epsilon^{\alpha}_{3}(u(\varepsilon)),$$

$$K^3_3(\varepsilon) = \epsilon^{-2}\varepsilon^3_3(u(\varepsilon)) = K^{33}(\varepsilon) = K_{33}(\varepsilon), \tag{3.14}$$

$$\sigma^{\alpha\beta} = \Lambda K^p_p(\varepsilon)g^{\alpha\beta} + 2G\epsilon^{\alpha\beta}(u(\varepsilon)), \tag{3.15}$$

$$\sigma^{\alpha 3}(\varepsilon) = \varepsilon^{-1}2GK^{\alpha 3}(u(\varepsilon)), \tag{3.16}$$

$$\sigma^{33}(\varepsilon) = \varepsilon^{-2}[\Lambda K^p_p(\varepsilon) + 2GK^3_3(\varepsilon)]; \tag{3.17}$$

and the scaled stresses are related to the real stresses through the relations

$$\sigma^{\alpha\beta}_{\varepsilon}(x^{\varepsilon}) = \frac{1}{\varepsilon}\sigma^{\alpha\beta}(\varepsilon)(x),$$

$$\sigma^{\alpha 3}_{\varepsilon}(x^{\varepsilon}) = \sigma^{\alpha 3}(\varepsilon)(x),$$

$$\sigma^{33}_{\varepsilon}(x^{\varepsilon}) = \varepsilon\sigma^{33}(\varepsilon)(x). \tag{3.18}$$

We shall denote by $K(\varepsilon)(v)$, formula (3.14) applied to any arbitrary vector field v in $IH^1_{\Gamma_0}$.

Remark. The relation (3.12) should be understood in the sense of trace (Adams, 1975) while the relation (3.13) should be understood in the sense of the trace operator defined by: Υ: $H(\text{div}, \Omega) \rightarrow IH^{-1/2}(\Gamma_1)$, where $H(\text{div}, \Omega) = \{(T^{ij});\ T^{ij}$ is in $L^2(\Omega)$ and $\text{div}\, T$ in $L^2(\Omega)\}$ (Ciarlet et al., 1989; Ciarlet, 1990).

4. The two-dimensional problem

We shall now deduce the limit two-dimensional equation of the sequence of variational eqn (3.10). To begin with, we first characterize the limit displacement.

Lemma 5. Let v be in $V_{kl} = \{v \text{ in } IH^1(\Omega),\ \epsilon_{i3}(v) = 0 \text{ in } \Omega;\ v = 0 \text{ on } \Gamma_0\}$, then there exist $\bar{v} = (\eta_\alpha, \eta_3)$ in $(H^1_{\gamma_0}(S))^2 \times H^2_{\gamma_0}(S)$ such that

$$v_\alpha(x^1, x^2, z) = [\mu^\nu_\tau \eta_\nu(x^1, x^2) - z\partial_\tau \eta_3(x^1, x^2)]\mu^\tau_\alpha, \quad \mu^\nu_\alpha = \delta^\nu_\alpha - zb^\nu_\alpha, \quad z = x^3,$$

$$v_3(x^1, x^2, z) = \eta_3(x^1, x^2). \tag{4.1}$$

Proof. Let $\varepsilon_{i3}(v) = 0$. Let $\bar{v} = (\bar{v}_\alpha, \bar{v}_3)$ be the components of $v = v_\alpha g^\alpha + v_3 a^3$ in the basis a^1, a^2, a^3. Then $v_\alpha = \mu^\nu_\alpha \bar{v}_\nu, v_3 = \bar{v}_3$ (see Section 2.3) and it follows that

$$\epsilon_{33}(v) = v_{3,3} = \bar{v}_{3,3} = 0 \tag{4.2}$$

$$\epsilon_{\alpha 3}(v) = \tfrac{1}{2}[\mu^\gamma_\alpha \bar{v}_{\gamma,3} + \bar{v}_{3,\alpha} + b^\gamma_\alpha \bar{v}_\gamma] = 0. \tag{4.3}$$

We deduce from eqn (4.2) that v_3 does not depend on $z = x^3$; $v_3 = 0$ on Γ_0 implies that $v_3 = \bar{v}_3$ is in $H^1(S)$ and $\bar{v}_3 = 0$ on γ_0. Multiplying $\epsilon_{\alpha 3}(v)$ by $(\mu^{-1})^\alpha_\lambda$ we obtain the equation

$$\bar{v}_{\lambda,3} + (\mu^{-1})^\alpha_\lambda b^\gamma_\alpha \bar{v}_\gamma = (\mu^{-1})^\alpha_\lambda \bar{v}_{3,\alpha}. \tag{4.4}$$

From the identity

$$(\mu^{-1})^\gamma_{\beta,3} = (\mu^{-1})^\lambda_\beta (\mu^{-1})^\alpha_\lambda b^\gamma_\alpha, \tag{4.5}$$

and multiplying eqn (4.4) by $(\mu^{-1})^\lambda_\beta$ we obtain the equivalent equation: find $\bar{v}_\lambda$ in $H^1(\Omega)$

$$((\mu^{-1})^\lambda_\beta \bar{v}_\lambda)_{,3} = -(\mu^{-1})^\lambda_\beta (\mu^{-1})^\alpha_\lambda \bar{v}_{3,\alpha}. \tag{4.6}$$

The eqns (4.3), (4.4) and (4.6) are all equivalent. It is easy to check that

$$\bar{v}_\gamma = \mu^\beta_\gamma \eta_\beta - z\bar{v}_{3,\gamma} \tag{4.7}$$

where η_β is in $H^1(S)$ is the solution. From eqn (4.7) and $v_\alpha = \mu^\beta_\alpha \bar{v}_\beta$ we deduce that $z\partial_\alpha v_3 = \mu^\beta_\alpha \eta_\beta - \bar{v}_\alpha$ is in $H^1(\Omega)$ and $\partial_\alpha v_3$ in $H^1(S)$.

Therefore v_3 is in $H^2(S)$ and $v_3 = 0$ on γ_0. From $v_\alpha = 0$ on γ_0 we deduce that $\bar{v}_\gamma = 0$ on γ_0 and $\eta_\beta = v_{3,\beta} = 0$ on γ_0. We conclude therefore that $(\eta_\alpha, \eta_3) \epsilon (H^1_{\gamma_0}(S))^2 \times H^2_{\gamma_0}(S)$. By simple computation we obtain eqn (4.1) which gives

$$v_\alpha = \mu^\gamma_\alpha \bar{v}_\gamma = \mu^\gamma_\alpha[\mu^\beta_\gamma \eta_\beta - z\bar{v}_{3,\gamma}] \quad \text{and} \quad v_3 = \bar{v}_3 = \bar{\eta}_3. \qquad \blacklozenge$$

Remark. It should be noted that V_{kl} is a closed subspace of $IH^1_{\Gamma_0}$. We also recall that all derivations used here should be understood in the sense of distribution and $v_\alpha = 0$ on Γ_0 as trace as was mentioned earlier. We are now able to characterize the limit displacement of the sequence of displacements $u(\varepsilon)$.

Theorem 3. As $\varepsilon \to 0$ the sequence $(u(\varepsilon))$, $\varepsilon > 0$ weakly converges in $IH^1_{\Gamma_0}(\Omega)$ to a displacement $u(0)$ in V_{kl} and the sequence $(K_{ij}(\varepsilon))$ defined in eqn (3.14) converges weakly in $L^2(\Omega)$ to K.

Proof. The proof is the same as in Ciarlet et al. (1989) and Ciarlet (1990). Using eqn (3.14), the scaled variational eqn (3.10) reads

$$\int_\Omega AK(\varepsilon) : K(\varepsilon)(v)\, d\Omega = L(v), \tag{4.8}$$

and letting $v = u(\varepsilon)$ we deduce from eqn (3.9) that there exists a constant C (independent on ε) such that

$$2G\|\epsilon(u(\varepsilon))\|^2 \leqslant \|K(\varepsilon)\|^2 \leqslant C\|u(\varepsilon)\|_{1,\Omega}; \tag{4.10}$$

the norms were defined in Section 2. We deduce by applying Korn's inequality that there exists a constant C such that

$$\|u(\varepsilon)\| \leqslant C, \quad \|\epsilon(u(\varepsilon))\| \leqslant C, \quad \|K(\varepsilon)\| \leqslant C; \tag{4.11}$$

consequently,

$$\|\epsilon_{\alpha 3}(\varepsilon)\| \leqslant C\varepsilon, \quad \|\epsilon_{33}(\varepsilon)\| \leqslant C\varepsilon^2; \tag{4.12}$$

and

$$u(\varepsilon) \overset{w}{\rightarrow} u(0) \quad \text{in } IH^1_{\Gamma_0}(\Omega), \tag{4.13}$$

$$K(\varepsilon) \overset{w}{\rightarrow} K \quad \text{in } L^2_S(\Omega). \tag{4.14}$$

From the semi-continuity of the norms we obtain,

$$u(0) \in V_{kl}. \tag{4.15}$$

◆

We deduce from Theorem 3 that there exist (ξ_α, ξ_3) in $(H^1_{\gamma_0}(S))^2 \times H^2_{\gamma_0}(S)$ such that

$$u_\alpha(0)(x^1, x^2, z) = [\mu^v_\tau \xi_v(x^1, x^2) - z\partial_\tau \xi_3(x^1, x^2)]\mu^\tau_\alpha,$$

$$= \xi_\alpha - z(\partial_\tau \xi_3 + 2b^\tau_\alpha \xi_\tau) + z^2(b^v_\tau b^\tau_\alpha \xi_\tau + b^\tau_\alpha \partial_\tau \xi_3); \tag{4.16}$$

$$u_3(0)(x^1, x^2, z) = \xi_3(x^1, x^2). \tag{4.17}$$

The limit displacement contains the classical Kirchhoff–Love displacement $\xi_\alpha - z\partial_\alpha \xi_3$ or that found in the Reissner–Mindlin model, $\xi_\alpha + z\psi_\alpha$ (here $\psi_\alpha = -(\partial_\alpha \xi_3 + 2b^\tau_\alpha \xi_\tau)$). The third term which is partly proportional to $\chi = h/R$ and χ^2 is implicitly assumed to be small in thin shells as a consequence of the hypotheses used in the framework.

Let $\bar{u} = (\xi_\alpha, \xi_3)$ be the components of $u(0)$ in the basis a^1, a^2, a^3, then

$$\epsilon_{\alpha\beta}(u(0)) = e_{\alpha\beta}(\bar{u}) - zk_{\alpha\beta}(\bar{u}) + z^2 Q_{\alpha\beta}(\bar{u}), \quad z = x^3; \tag{4.18}$$

where $e_{\alpha\beta}(\bar{u})$, $k_{\alpha\beta}(\bar{u})$, $Q_{\alpha\beta}(\bar{u})$ defined in Section 2 are, respectively, the membrane deformation tensor, the change of curvature tensor and the change in the third fundamental form as was previously announced. This last term usually disappears in the classical framework.

In order to characterize the limit variational equation, we first recall a lemma due to Ciarlet (1990) which considerably simplifies the proof as compared to that given in Ciarlet et al. (1989).

Lemma 6. Let $w \in L^2(\Omega)$ be such that

$$\int_\Omega wv_{,3}\, dx = 0 \quad \text{for all } v \quad \text{in } C^\infty(\bar{\Omega}), \quad v = 0 \quad \text{on } \gamma x[-1,1], \gamma = \partial S,$$

where x is the coordinate system, then $w = 0$. ◆

Theorem 4. As $\varepsilon \to 0$ the components of the weak limit symmetric tensor K satisfy:

$$K_{\alpha\beta} = \epsilon_{\alpha\beta}(u(0)), \tag{4.19}$$

$$K_{\tau 3} = K^{\tau 3} = 0, \tag{4.20}$$

$$K_{33} = K^{33} = K_3^3 = -\frac{\Lambda}{\Lambda + 2G}\epsilon_\alpha^\alpha(u(0)). \tag{4.21}$$

Proof. We obviously obtain eqn (4.19) since $u(\varepsilon) \xrightarrow{w} u(0)$ in $IH^1_{\Gamma_0}(\Omega)$. Let $v = (v^1, v^2, v^3)$ be such that $v \,\varepsilon\, C^\infty(\bar{\Omega})$, $v = 0$ on $\gamma\ x\ [-1, 1]$ and $v^3 = 0$. Multiplying eqn (3.11) by εv and integrating, we obtain, from Green's formula

$$2G\int_\Omega K^{\alpha 3}(\varepsilon)v_{\alpha/3}\,\mathrm{d}\Omega = \varepsilon L(v) - \varepsilon\int_\Omega \Lambda K_\rho^\rho(\varepsilon)\epsilon_\alpha^\alpha(v) + 2\Lambda K_{\alpha\beta}(\varepsilon)\epsilon^{\alpha\beta}(v)\,\mathrm{d}\Omega,$$

$$\xrightarrow{w} 2G\int_\Omega [K^{\alpha 3}(\mu)_\alpha^v]\bar{v}_{,3}\rho\sqrt{a}\,\mathrm{d}x = 0 \tag{4.22}$$

because of the uniform boundedness of $L(v)$ and the weak convergence of $K(\varepsilon)$, for each fixed v. We deduce from Lemma 6 that

$$K^{\alpha 3}(\mu)_\alpha^v\rho\sqrt{a} = 0 \tag{4.23}$$

and multiplying by $(\mu^{-1})_v^\tau/\rho\sqrt{a}$ implies that $K^{\tau 3} = 0$ and from Section 2 we also deduce that $K_{\tau 3} = 0$. Similarly by letting $v = (0, 0, v^3)$ and multiplying both sides by ε^2, passing to the limit, since $K^{\alpha 3}(\varepsilon) \to 0$, we obtain

$$\int_\Omega [\Lambda K_\alpha^\alpha + (\Lambda + 2G)K_3^3]v_{3,3}\,\mathrm{d}\Omega = 0 \quad \text{or } K_3^3 = -\frac{\Lambda}{\Lambda + 2G}K_\alpha^\alpha \tag{4.24}$$

and eqn (4.21) is thus proved since $K_\alpha^\alpha = \epsilon_\alpha^\alpha(u(0))$. ◆

Theorem 5. Let the limit displacement $u(0)$ be defined through $\bar{u} = (\xi_\alpha, \xi_3)$ in $(H^1_{\gamma_0}(S))^2 \times H^2_{\gamma_0}(S)$.

(i) As $\varepsilon \to 0$ the sequence of variational problem (3.10) converges to a well defined variational problem on $\bar{u} = (\xi_\alpha, \xi_3)$:

$$\int_S \left[\int_{-1}^1 \frac{2G\Lambda}{\Lambda + 2G} g^{\alpha\gamma}g^{\beta\delta}\epsilon_{\alpha\gamma}(u(0))\epsilon_{\beta\delta}(v) + 2Gg^{\alpha\delta}g^{\beta\gamma}\epsilon_{\alpha\beta}(u(0))\epsilon_{\delta\gamma}(v)\right]\rho\,\mathrm{d}z\,\mathrm{d}S - \bar{L}(\bar{v}) = 0, \tag{4.25}$$

for $v = (v^1, v^2, v^3)$ defined by $\bar{v} = (\eta_\alpha, \eta_3)$ in $(H^1_{\gamma_0}(S))^2 \times H^2_{\gamma_0}(S))$;

(ii) The whole family $(u(\varepsilon))$ converges weakly to $u(0)$.

Proof. Let $v \in V_{kl}$ be used in eqn (3.10). We obtain

$$\int_{\Omega} \Lambda(K_{\tau}^{\tau}(\varepsilon)+K_3^3(\varepsilon))\epsilon_{\alpha}^{\alpha}(v)+2GK_{\alpha\beta}(\varepsilon)\epsilon^{\alpha\beta}(v)\,\mathrm{d}\Omega = L(v); \tag{4.26}$$

passing to the limit and using Theorem 4, we obtain

$$\int_{\Omega} \Lambda\left(1-\frac{\Lambda}{\Lambda+2G}\right)\epsilon_{\tau}^{\tau}(u(0))\epsilon_{\alpha}^{\alpha}(v)+2G\epsilon_{\alpha\beta}(u(0))\epsilon^{\alpha\beta}(v)\,\mathrm{d}\Omega = L(v). \tag{4.27}$$

We next apply formulae found in Section 2.2 on ϵ_{τ}^{τ} and $\epsilon^{\alpha\beta}$ to obtain eqn (4.25).

Now the left-hand side of eqn (3.25) can be written as

$$\int_{\Omega} N\epsilon(u(0))\colon \epsilon(v)\,\mathrm{d}\Omega = L(v), \tag{4.28}$$

where

$$N^{ijkl}(x) = \bar{\Lambda}g^{ij}g^{kl}+G(g^{ik}g^{jl}+g^{il}g^{jk}), \quad \bar{\Lambda} = \frac{2G\Lambda}{\Lambda+2G}, \quad G > 0 \tag{4.29}$$

and consequently there exists a constant $C > 0$ such that for all (ϵ_{ij}), $\epsilon_{i3} = 0$,

$$C\int_{\Omega} \epsilon_{\alpha\beta}\epsilon_{\alpha\beta}\,\mathrm{d}\Omega \leqslant \int_{\Omega} N\epsilon : \epsilon\,\mathrm{d}\Omega. \tag{4.30}$$

We shall now show that the left-hand side of eqn (4.25) which is a continuous bilinear form in $(H_{\gamma_0}^1(S))^2 \times H_{\gamma_0}^2(S)$ is elliptic. Let $u \in V_{kl}$, we deduce from eqn (4.30) that

$$C\int_{\Omega} \epsilon_{\alpha\beta}(u)\epsilon_{\alpha\beta}(u)\rho\,\mathrm{d}S\,\mathrm{d}z \leqslant \int_{\Omega} N\epsilon(u)\colon \epsilon(u)\,\mathrm{d}\Omega \tag{4.31}$$

and from eqn (4.18) that there exists a constant $C > 0$ such that

$$C\int_S\int_{-1}^{1} \epsilon_{\alpha\beta}(u)\epsilon_{\alpha\beta}(u)\,\mathrm{d}z\,\mathrm{d}S \leqslant C\int_S \left(2e\colon e+\frac{4}{3}e\colon Q+\frac{2}{3}k\colon +\frac{2}{5}Q\colon Q\right)\mathrm{d}S \leqslant \int_{\Omega} N\epsilon(u)\colon \epsilon(u)\,\mathrm{d}\Omega \tag{4.32}$$

since there exists a constant $\rho_0 > 0$ such that $\rho \geqslant \rho_0$ (Section 2). We also have

$$\left|\frac{4}{3}e\colon Q\right| = \left|\frac{2\sqrt{2}}{\sqrt{3}}e\colon \frac{2}{\sqrt{3}\sqrt{2}}Q\right| \leqslant \frac{4}{3}e\colon e + \frac{1}{3}Q\colon Q \tag{4.33}$$

by applying Hölder's inequality $|ab| \leqslant (a^2+b^{2)}/2$. We therefore obtain the inequality

$$\frac{2}{3}C\int_S (e : e+k : k)\,\mathrm{d}S \leqslant C\int_S \left(\frac{2}{3}e : e+\frac{2}{3}k : k+\frac{1}{15}Q : Q\right)\mathrm{d}S \leqslant \int_{\Omega} N\epsilon(u) : \epsilon(u)\,\mathrm{d}\Omega. \tag{4.34}$$

In conclusion we find that there exists a constant $C > 0$ such that

$$C\int_S (e(\bar{u}):e(\bar{u})+k(\bar{u}):k(\bar{u}))\,\mathrm{d}S \leqslant \int_\Omega N\epsilon(u(0)):\epsilon(u(0))\,\mathrm{d}\Omega \tag{4.35}$$

and the coercitivity is established by applying Lemma 2.

The right-hand side of eqn (4.25) is obviously a continuous linear form on $\bar{v} = (\eta_\alpha, \eta_3)$. From the elipticity established above, we deduce from Lax Milgram that the limit variational eqn (4.25) has a unique solution in $(H^1_{\gamma_0}(S))^2 \times H^2_{\gamma_0}(S)$.

The uniqueness of the weak limit also implies that the whole family $(u(\varepsilon))$ converges to the unique limit. ◆

Remark. We can also rewrite $\rho N^{ijkl}(x)$ found in the left-hand side of eqn (4.25) by using the expression of $(g^{\alpha\beta}) = (\mu^{-1})^\alpha_\gamma(\mu^{-1})^\beta_\tau \alpha^{\gamma\tau}$. If the exact expression of $(\mu^{-1})^\alpha_\tau$ is computed and used in these formulas and if it is assumed that $\rho \cong 1$ and $(\mu^{-1})^\alpha_\tau = \delta^\alpha_\tau$, these formulas will automatically lead to the modulus tensor found in the theory of thin shells. If Q is also neglected then eqn (4.25) will yield the usual variational equation found in thin shells. If different test functions such as

$$v_\alpha = \xi_\alpha - z\partial_\alpha\xi_3, \quad v_3 = \xi_3; \quad \text{or } v_\alpha = \xi_\alpha - z(\partial_\alpha\xi_3 + 2b^\tau_\alpha\xi_\tau), \quad v_3 = \xi_3; \tag{4.36}$$

or if the crucial hypotheses (3.5) is modified, for example by, $(b^\varepsilon_{\alpha\beta}) = \varepsilon^t(b_{\alpha\beta})$, $(b^\varepsilon)^\alpha_\beta = \varepsilon^{-t}(b)^\alpha_\beta$; different limit models will be obtained. These investigations will provoke unnecessarily lengthiness in this paper. These topics will be widely treated in Nzengwa (1998b).

Theorem 6. As $\varepsilon \to 0$ the whole family $(u(\varepsilon))_{\varepsilon>0}$ converges strongly in $IH^1_{\Gamma_0}(\Omega)$ and the whole family $(K(\varepsilon))_{\varepsilon>0}$ converges strongly in $L^2_S(\Omega)$.

Proof. The proof is the same as in Ciarlet et al. (1989) or Ciarlet (1990). We first observe that the limit displacement $u(0)$ and the tensor K satisfy the equations:

$$\int_\Omega AK : K\,\mathrm{d}\Omega = L(u(0)) \quad \text{and} \quad \int_\Omega AK(\varepsilon) : K(\varepsilon)\,\mathrm{d}\Omega = L(u(\varepsilon)). \tag{4.37}$$

From eqn (3.10) or equivalently eqns (4.9) and (4.37) we obtain

$$\begin{aligned} 2G\|K-K(\varepsilon)\|^2 &\leqslant \int_\Omega A(K(\varepsilon)-K):(K(\varepsilon)-K)\,\mathrm{d}\Omega \\ &= \int_\Omega AK:(K-2K(\varepsilon))\,\mathrm{d}\Omega + \int_\Omega AK(\varepsilon):K(\varepsilon)\,\mathrm{d}\Omega \\ &= \int_\Omega AK:(K-2K(\varepsilon))\,\mathrm{d}\Omega + L(u(\varepsilon)). \end{aligned} \tag{4.38}$$

From the weak convergence of $K(\varepsilon)$ to K and $u(\varepsilon)$ to $u(0)$, we deduce from eqn (4.37) that

$$\|K-K(\varepsilon)\| \overset{\varepsilon}{\to} 0. \tag{4.39}$$

By using the definition of $K_{\alpha\beta}$, eqn (4.39) also implies that $\|\epsilon_{\alpha\beta}(u(0)) - \epsilon_{\alpha\beta}(u(\varepsilon))\| \overset{\varepsilon}{\to} 0$ and the strong convergence of $u(\varepsilon)$ is deduced by applying Korn's inequality since $\|\epsilon_{i3}(u(0)) - \epsilon_{i3}(u(\varepsilon))\|$ also converges to 0. ◆

We shall denote by $H^{-1}(S)$ and $H^{-2}(S)$ the dual space of $H^1_0(S)$ and $H^2_0(S)$, respectively. We shall further consider the spaces $H^1(-1, 1; H^{-1}(S))$ and $H^2(-1, 1; H^{-2}(S))$.

We recall that $d\Omega = \rho \, dS \, dz$. The usual compact injections (denoted $\xrightarrow{i}$) also hold because of the boundedness of ρ:

$$L^2(\Omega) \xrightarrow{i} L^2(-1,1;L^2(S)) \xrightarrow{i} L^2(-1,1;H^{-1}(S)) \xrightarrow{i} L^2(-1,1;H^{-2}(S)). \tag{4.40}$$

We shall also use Gronwall's lemma: (Cartan, 1977; Brezis, 1973, 1983; Crouzeix and Mignot, 1983).

Lemma 6. Let there be given a function X in $L^1([a, b]; IR_+)$ such that dX/dt be in $L^1([a, b]; IR)$ and

$$\frac{d}{dt}X(t) \leqslant CX(t)+C, \tag{4.41}$$

then

$$X(t) \leqslant C\exp(Ct) \tag{4.42}$$

and X is consequently bounded in $[a, b]$. ◆

Details on the above spaces and the lemma can be found in Brezis (1973). We can now compute the shear stresses $\sigma^{\alpha 3}$, σ^{33}.

Theorem 7. Let f^α be in $L^2(\Omega)$, f^3 in $H^1(\Omega)$, g^i in $L^2(\Gamma_1)$, then as $\varepsilon \to 0$ the scaled stresses $\sigma^{\alpha\beta}(\varepsilon)$, $\sigma^{\alpha 3}(\varepsilon)$, $\sigma^{33}(\varepsilon)$ defined by eqns (3.15)–(3.17) converge strongly as follow:

$$\sigma^{\alpha\beta}(\varepsilon) \to \sigma^{\alpha\beta} = \bar{\Lambda}\epsilon^\tau_\tau(u(0))g^{\alpha\beta}+2G\epsilon^{\alpha\beta}(u(0)) \quad \text{in } L^2(\Omega); \tag{4.43}$$

$$\sigma^{\alpha 3}(\varepsilon) \to \sigma^{\alpha 3} \quad \text{in } H^1(-1,1;H^{-1}(S)); \tag{4.44}$$

$$\sigma^{33}(\varepsilon) \to \sigma^{33} \quad \text{in } H^2(-1,1;H^{-2}(S)); \tag{4.45}$$

$\sigma^{\alpha 3}$, σ^{33} are solutions to the equations

$$\sigma^{\alpha 3} \in H^1(-1,1;H^{-1}(S)); \tag{4.46}$$

$$\frac{d\sigma^{\alpha 3}}{dz}+2\Gamma^\alpha_{\beta 3}\sigma^{\beta 3}+\Gamma^\beta_{\beta 3}\sigma^{\alpha 3} = -(\sigma^{\alpha\beta}_{,\beta}+\Gamma^\alpha_{\beta\tau}\sigma^{\tau\beta}+\Gamma^\beta_{\beta\tau}\sigma^{\alpha\tau})-f^\alpha; \tag{4.47}$$

$$\sigma^{\alpha 3}(-1) = -\bar{h}^\alpha_-; \tag{4.48}$$

$$\sigma^{\alpha 3}(1) = \bar{h}^\alpha_+; \tag{4.49}$$

$$\sigma^{33} \in H^2(-1,1;H^{-2}(S)); \tag{4.50}$$

$$\frac{d\sigma^{33}}{dz}+\Gamma^\alpha_{\alpha 3}\sigma^{33} = -(\sigma^{3\alpha}_{,\alpha}+\Gamma^3_{\alpha\tau}\sigma^{\tau\alpha}+\Gamma^\beta_{\beta\tau}\sigma^{3\tau})-f^3; \tag{4.51}$$

$$\sigma^{33}(-1) = -\bar{h}^3_-; \tag{4.52}$$

$$\sigma^{33}(1) = \bar{h}^3_+. \tag{4.53}$$

Proof. From the definition of $\sigma^{\alpha\beta}(\varepsilon)$ eqn (3.15), the convergence of $K_3^3(\varepsilon)$ (Theorem 4) and the strong convergence of $K_{ij}(\varepsilon)$ (Theorem 6), we deduce eqn (4.43). From the existence of the solution $u(\varepsilon)$ (Theorem 2) and the definitions (3.15)–(3.17) it follows that

$$\sigma^{\alpha 3}(\varepsilon) \quad \text{is in } L^2(\Omega) \overset{i}{\to} L^2(-1,1;H^{-1}(S)) \quad \text{and} \quad \sigma^{33}(\varepsilon) \quad \text{is in } L^2(\Omega) \overset{i}{\to} L^2(-1,1;H^{-2}(S)).$$

From eqns (3.11)–(3.13) we have

$$\frac{\mathrm{d}\sigma^{\alpha 3}(\varepsilon)}{\mathrm{d}z} + 2\Gamma^{\alpha}_{\beta 3}\sigma^{\beta 3}(\varepsilon) + \Gamma^{\beta}_{\beta 3}\sigma^{\alpha 3}(\varepsilon) = -(\sigma^{\alpha\beta}_{,\beta}(\varepsilon) + \Gamma^{\alpha}_{\beta\tau}\sigma^{\tau\beta}(\varepsilon) + \Gamma^{\beta}_{\beta\tau}\sigma^{\alpha\tau}(\varepsilon)) - f^{\alpha}; \tag{4.54}$$

$$\sigma^{\alpha 3}(\varepsilon)(-1) = -\bar{h}^{\alpha}_{-}; \tag{4.55}$$

$$\frac{\mathrm{d}\sigma^{33}(\varepsilon)}{\mathrm{d}z} + \Gamma^{\alpha}_{\alpha 3}\sigma^{33}(\varepsilon) = -(\sigma^{3\alpha}_{,\alpha}(\varepsilon) + \Gamma^{3}_{\alpha\tau}\sigma^{\tau\alpha}(\varepsilon) + \Gamma^{\beta}_{\beta\tau}\sigma^{3\tau}(\varepsilon)) - f^{3}; \tag{4.56}$$

$$\sigma^{33}(\varepsilon)(-1) = -\bar{h}^{3}_{-}. \tag{4.57}$$

Equations (4.54)–(4.57) are differential equations with Lipschitz continuous vector field. They therefore have solutions defined for all z in $[-1, 1]$ which satisfy the initial conditions (4.55) and (4.57). The right-hand of eqn (4.54) is uniformly bounded because of eqn (4.43). Multiplying eqn (4.54) by $\sigma^{\alpha 3}(\varepsilon)$, integrating over Ω and applying Hölder's inequality, we deduce that

$$\sum_{\alpha} \frac{\mathrm{d}}{\mathrm{d}z} \|\sigma^{\alpha 3}(\varepsilon)\|^2 \leqslant C \sum_{\alpha} \|\sigma^{\alpha 3}(\varepsilon)\|^2 + C \tag{4.58}$$

and from Lemma 6, $\sigma^{\alpha 3}(\varepsilon)$ is bounded in $L^2(\Omega)$; consequently converges weakly in $L^2(\Omega)$ to $\sigma^{\alpha 3}$ and strongly converges in $L^2(-1, 1; H^{-1}(S))$. We deduce from this convergence that eqns (4.47) and (4.48) are satisfied.

Multiplying eqn (4.56) by $\sigma^{33}(\varepsilon)$, integrating over Ω and applying Hölder's inequality we deduce in the same way through Lemma 6 and the boundedness of $\sigma^{\alpha 3}(\varepsilon)$ that $\sigma^{33}(\varepsilon)$ is bounded in $L^2(\Omega)$ and consequently converges weakly to σ^{33} in $L^2(\Omega)$ and strongly in $L^2(-1, 1; H^{-2}(S))$. Similarly we deduce that eqns (4.51) and (3.52) are satisfied.

The eqn (4.47) also implies that $\mathrm{d}\sigma^{\alpha 3}/\mathrm{d}z$ is in $L^2(-1, 1; H^{-1}(S))$ and therefore $\sigma^{\alpha 3}$ is in $H^1(-1, 1; H^{-1}(S))$. Similarly we deduce from eqn (4.51) that $\sigma^{33}(\varepsilon)$ is in $H^2(-1, 1; H^{-2}(S))$.

The eqns (4.47) and (4.51) are also equivalent to

$$\sigma^{\alpha\beta}_{/\beta} + \sigma^{\alpha 3}_{/3} + f^{\alpha} = 0 \quad \text{in } \Omega; \tag{4.59}$$

$$\sigma^{3\beta}_{/\beta} + \sigma^{33}_{/3} + f^{3} = 0 \quad \text{in } \Omega. \tag{4.60}$$

Let $v = (v_\alpha, v_3)$, v_α in $H^1_0(S)$; $v_3 = z\bar{v}_3$, $\bar{v}_3$ in $H^2_0(S)$ be used in the scaled equation (3.10). Then $\epsilon_{33}(v) = \bar{v}_3$ and passing to the limit, we obtain

$$\int_{\Omega} (\sigma^{\alpha\beta}\epsilon_{\alpha\beta}(v) + \sigma^{\alpha 3}\epsilon_{\alpha 3}(v) + \sigma^{3\beta}\epsilon_{3\beta}(v) + \sigma^{33}\epsilon_{33}(v))\,\mathrm{d}\Omega$$

$$= \int_{\Omega} f^{\alpha} v_{\alpha}\,\mathrm{d}\Omega + \int_{S} (\bar{h}^{\alpha}_{+} + \bar{h}^{\alpha}_{-}) v_{\alpha}\,\mathrm{d}S + \int_{\Omega} f^{3} v_3\,\mathrm{d}\Omega + \int_{S} (\bar{h}^{3}_{+} - \bar{h}^{3}_{-})\bar{v}_3\,\mathrm{d}S. \tag{4.61}$$

Multiplying eqn (4.59) by v_α and eqn (4.60) by v_3 and integrating, from Green's formula and eqn (4.61), we obtain

$$\int_S (\sigma^{\alpha 3}(1)-\sigma^{\alpha 3}(-1))v_\alpha \, \mathrm{d}S + \int_S (\sigma^{33}(1)+\sigma^{33}(-1))\bar{v}_3 \, \mathrm{d}S$$

$$= \int_S (\bar{h}^\alpha_+ + \bar{h}^\alpha_-)v_\alpha \, \mathrm{d}S + \int_S (\bar{h}^3_+ - \bar{h}^3_-)\bar{v}_3 \, \mathrm{d}S. \quad (4.62)$$

Since v_α and $\bar{v}_3$ are arbitrary test functions, we deduce that

$$\sigma^{\alpha 3}(1) = \sigma^{\alpha 3}(-1) + \bar{h}^\alpha_+ + \bar{h}^\alpha_- = \bar{h}^\alpha_+; \quad (4.63)$$

$$\sigma^{33}(1) = -\sigma^{33}(-1) + \bar{h}^3_+ - \bar{h}^3_- = \bar{h}^3_+; \quad (4.64)$$

and eqns (4.49) and (4.53) are satisfied. ◆

Remark. Though the convergence of the shear stresses $\sigma^{\alpha 3}(\varepsilon)$ and $\sigma^{33}(\varepsilon)$ are strong in the spaces $H^1(-1, 1; H^{-1}(S))$ and $H^2(-1, 1; H^{-2}(S))$, respectively, they are in fact very 'weak'. Because σ^{33} belongs to $H^2(-1, 1; H^{-2}(S))$ it should be noted that σ^{33} can be a localized stress and such a stress distribution is of paramount interest in junctions of multi-structures.

From the convergence established and the relation between the scaled and de-scaled stresses (3.18) we deduce that the real stress

$$\sigma^{33}(x^\varepsilon) \to 0 \quad \text{as} \quad \varepsilon \to 0 \quad \text{in } L^2(\Omega). \quad (4.65)$$

Unlike in plate theory (Destuynder, 1986) the exact formula of $\sigma^{\alpha 3}$ and σ^{33} cannot be expressed but these stresses can be computed numerically by approximating eqns (4.47) and (4.48) and (4.51) and (4.52).

In the exact two-dimensional problem (4.25), one will have to compute integrals of the form

$$\bar{N}_n = (\bar{N}_n^{\alpha\beta\gamma\delta}(\bar{x})) = \left(\int_{-1}^{1} \rho(\bar{x}, z) z^n N^{\alpha\beta\gamma\delta}(\bar{x}, z) \, \mathrm{d}z\right), \quad n = 0, 1, 2, 3, 4; \quad \bar{x} = (x^1, x^2); \quad (4.66a)$$

because of the particular form in eqn (4.18) of $\epsilon_{\alpha\beta}(v)$, v in V_{kl}. If the exact expression of $(\mu^{-1})^\alpha_\beta$ (see Section 2) is used in eqn (4.36), then in eqn (4.66) one will have to compute integrals of rational polynomials $\Sigma_{m=0}^{8} p_m(z)/\bar{\rho}(z)$ where the degree of $\bar{\rho}(z) = (\rho(\bar{x}, z))^3$ is 6 and that of $p_m(z)$ is m.

The exact expression is complex but not necessary since numerical integration can easily be implemented.

Let $\epsilon_{\alpha\beta}(u(0)) = e_{\alpha\beta}(\bar{u}) - zk_{\alpha\beta}(\bar{u}) + z^2 Q_{\alpha\beta}(\bar{u})$, $z = x^3$ and $\epsilon_{\alpha\beta}(v) = e_{\alpha\beta}(\bar{v}) - zk_{\alpha\beta}(\bar{v}) + z^2 Q_{\alpha\beta}(\bar{v})$ and let us denote $\epsilon(u(0)) = e - zk + z^2 Q$, $\epsilon(v) = \bar{e} - z\bar{k} + z^2\bar{Q}$, where $\bar{u} = (\xi_\alpha, \xi_3)$, $\bar{v} = (\eta_\alpha, \eta_3)$ is in $(H^1_{\gamma_0}(S))^2 \times H^2_{\gamma_0}(S)$; then, by using the de-scaling relations

$$\xi^\varepsilon_\alpha(x^1, x^2) = \varepsilon^2 \xi_\alpha(x^1, x^2); \quad \xi^\varepsilon_3(x^1, x^2) = \varepsilon \xi_3(x^1, x^2) \quad \text{for all } (x^1, x^2) \in \bar{S};$$

$$b^\varepsilon_{\alpha\beta} = \varepsilon b_{\alpha\beta}, \quad (b^\varepsilon) = \frac{1}{\varepsilon}(b)^\alpha_\beta, \quad z^\varepsilon = \varepsilon z, \quad z = x^3;$$

and replacing $\bar{N}_n$ by

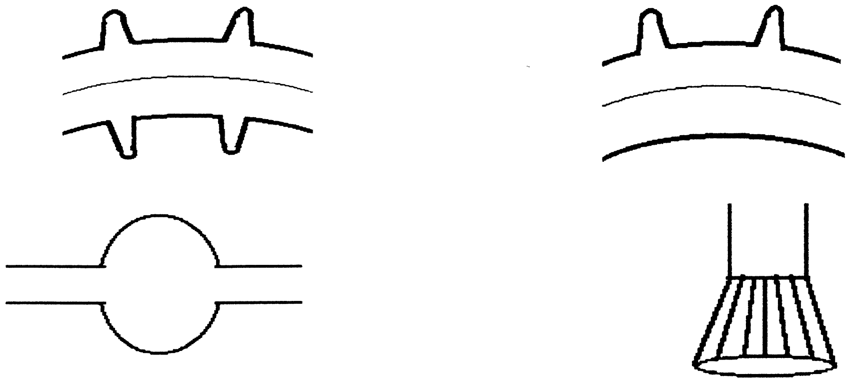

Fig. 4. Stiffened shells.

$$\bar{N}_n^\varepsilon = (\bar{N}_n^{\varepsilon,\alpha\beta\gamma\delta}) = \left(\int_{-\varepsilon}^{\varepsilon} \rho(\bar{x}, z) z^n N^{\alpha\beta\gamma\delta}(\bar{x}, z)\, \mathrm{d}z\right), \quad n = 0, 1, 2, 3, 4; \quad \bar{x} = (x^1, x^2), \tag{4.66b}$$

the two-dimensional variational equation of the real shell deduced from eqn (4.25) now reads:

$$\text{find}(\xi_\alpha^\varepsilon, \xi_3^\varepsilon) \in (H_{\gamma_0}^1(S))^2 \times H_{\gamma_0}^2(S) \text{ such that}$$

$$\int_S (\bar{N}_0^\varepsilon e\colon \bar{e} + \bar{N}_2^\varepsilon k\colon \bar{k} + \bar{N}_2^\varepsilon e\colon \bar{Q} + \bar{N}_2^\varepsilon Q\colon \bar{e} + \bar{N}_4^\varepsilon Q\colon \bar{Q} - \bar{N}_1^\varepsilon e\colon \bar{k} - \bar{N}_1^\varepsilon k\colon \bar{e} - \bar{N}_3^\varepsilon k\colon \bar{Q}$$
$$- \bar{N}_3^\varepsilon Q\colon \bar{k})\, \mathrm{d}S = \bar{L}_\varepsilon(\bar{v}). \tag{4.67}$$

The exact expression of $\bar{L}_\varepsilon$ can be deduced easily from $\bar{L}$. The variational eqn (4.67) is well defined and the corresponding boundary-value problem can be deduced as shown below. Finite element methods can then be implemented to compute the exact two-dimensional displacements $(\xi_\alpha^\varepsilon, \xi_3^\varepsilon)$, the three-dimensional displacements $u^\varepsilon = (u_i^\varepsilon)$, the in-plane stresses $\sigma_\varepsilon^{\alpha\beta}$ [see scaled formula (4.42)] and shear stresses σ_ε^{i3} as indicated.

We recall that if the usual assumptions on thin shells are admitted, except the first two terms of the left-hand side of eqn (4.67), all the others will disappear and the right-hand side will have to be modified because of the form of the limit displacement. Because of the Taylor expansion of $\rho(\mu^{-1})_\beta^\alpha$, we shall be interested in the best first-order equations which have similarities with equations widely used in engineering.

5. The best first order two-dimensional problem

The Taylor expansion of $(\mu^{-1})_\beta^\alpha$ given in Lemma 1 can also be written as

$$(\mu^{-1})_\beta^\alpha = \delta_\beta^\alpha + z b_\beta^\alpha + \sum_{n \geqslant 2} z^n (b^n)_\beta^\alpha, \quad z = x^3.$$

Recall that $(g^{\alpha\beta}) = (\mu^{-1})^{\alpha}_{\gamma}(\mu^{-1})^{\beta}_{\tau}a^{\gamma\tau}$ and $\rho(\bar{x}, z) = 1 - zb^{\alpha}_{\alpha} + z^2(b^1_1 b^2_2 - b^1_2 b^2_1)$. Using these expressions, the left-hand side of the two-dimensional scaled variational eqn (4.25) or (4.67) takes the form

$$A_1(\bar{u}, \bar{v}) + zb^{\alpha}_{\tau}B^{\tau}_{\alpha}(\bar{u}, \bar{v}) = \bar{L}(\bar{v}) = \int_S (p^{\alpha}\eta_{\alpha} + p^3\eta_3)\,\mathrm{d}S + \int_{\gamma_1} (q^{\alpha}\eta_{\alpha} + q^3\eta_3)\,\mathrm{d}\gamma + \int_{\gamma_1} m^{\alpha}\theta_{\alpha}\,\mathrm{d}\gamma \tag{5.1}$$

where A_1 and B^{τ}_{α} are continuous bilinear forms; with

$$\begin{aligned} A_1(\bar{u}, \bar{v}) &= \frac{2E}{1-\bar{v}^2}\int_S [(1-\bar{v})e^{\alpha\beta}(\bar{u}) + \bar{v}e^{\mu}_{\mu}(\bar{u})a^{\alpha\beta}]e_{\alpha\beta}(\bar{v})\,\mathrm{d}s \\ &+ \frac{2E}{3(1-\bar{v}^2)}\int_S [(1-\bar{v})K^{\alpha\beta}(\bar{u}) + \bar{v}K^{\mu}_{\mu}(\bar{u})a^{\alpha\beta}]K_{\alpha\beta}(\bar{v})\,\mathrm{d}s \\ &+ \boldsymbol{\frac{2E}{3(1-\bar{v}^2)}\int_S [(1-\bar{v})e^{\alpha\beta}(\bar{u}) + \bar{v}e^{\mu}_{\mu}(\bar{u})a^{\alpha\beta}]Q_{\alpha\beta}(\bar{v})\,\mathrm{d}s} \\ &+ \boldsymbol{\frac{2E}{3(1-\bar{v}^2)}\int_S [(1-\bar{v})Q^{\alpha\beta}(\bar{u}) + \bar{v}Q^{\mu}_{\mu}(\bar{u})a^{\alpha\beta}]e_{\alpha\beta}(\bar{v})\,\mathrm{d}s} \\ &+ \boldsymbol{\frac{5E}{5(1-\bar{v}^2)}\int_S [(1-\bar{v})Q^{\alpha\beta}(\bar{u}) + \bar{v}Q^{\mu}_{\mu}(\bar{u})a^{\alpha\beta}]Q_{\alpha\beta}(\bar{v})\,\mathrm{d}s} \end{aligned} \tag{5.2}$$

$$\bar{L}(\bar{v}) = \int_S [p^{\alpha}\eta_{\alpha} + p^3\eta_3]\,\mathrm{d}s + \int_{\gamma_1} [q^{\alpha}\eta_{\alpha} + q^3\eta_3]\,\mathrm{d}s + \int_{\gamma_1} m^{\alpha}\theta_{\alpha}\,\mathrm{d}\gamma; \tag{5.3}$$

$$p^{\alpha} = \int_{-1}^{1} f^{\tau}w^{\alpha}_{\tau}(z)\,\mathrm{d}z + \boldsymbol{h^{\tau}_{+}w^{\alpha}_{\tau}(1) + \bar{h}^{\tau}_{-}w^{\alpha}_{\tau}(-1)}; \quad w^{\alpha}_{\tau}(z) = \delta^{\alpha}_{\tau} - zb^{\alpha}_{\tau} + \boldsymbol{z^2 b^{\alpha}_{\nu}b^{\nu}_{\tau}}, \tag{5.4}$$

$$p^3 = \int_{-1}^{1} f^3\,\mathrm{d}z + \bar{h}^3_{+} + \bar{h}^3_{-} - \partial_{\tau}\boldsymbol{[f^{\alpha}\bar{w}^{\tau}_{\alpha}(z)\,\mathrm{d}z + \bar{h}^{\alpha}_{+}\bar{w}^{\tau}_{\alpha}(1) + \bar{h}^{\alpha}_{-}w^{\tau}_{\alpha}(-1)]}; \quad \bar{w}^{\alpha}_{\tau}(z) = (-z\delta^{\tau}_{\alpha} + \boldsymbol{z^2 b^{\tau}_{\alpha}}), \tag{5.5}$$

$$q^{\alpha} = \int_{-1}^{1} \bar{h}^{\alpha}\,\mathrm{d}z - \int_{-1}^{1} \boldsymbol{zb^{\alpha}_{\gamma}\bar{h}^{\gamma}\,\mathrm{d}z}, \tag{5.6}$$

$$q^3 = \int_{-1}^{1} \bar{h}^3\,\mathrm{d}z, \tag{5.7}$$

$$m^{\alpha} = \int_{-1}^{1} z\bar{h}^{\alpha}\,\mathrm{d}z - \int_{-1}^{1} \boldsymbol{z^2 b^{\alpha}_{\gamma}\bar{h}^{\gamma}\,\mathrm{d}z}, m = m^{\alpha}a_{\alpha} = m_{\alpha}a^{\alpha}; \tag{5.8}$$

$$\theta_{\alpha} = -(b^{\gamma}_{\alpha}\eta_{\gamma} + \nabla_{\alpha}\eta_3), \tag{5.9}$$

$$\bar{u} = (\xi_{\alpha}, \xi_3) \quad \text{is in } (H^1_{\gamma_0}(S))^2 \times H^2_{\gamma_0}(S), \quad \bar{v} = (\eta_{\alpha}, \eta_3) \quad \text{is in } (H^1_{\gamma_0}(S))^2 \times H^2_{\gamma_0}(S). \tag{5.10}$$

All terms in bold are new. In the forces they are proportional to $\chi = h/R$ or χ^2 because of the particular form of the displacement. It should be observed that the five terms found in $A_1(\bar{u},\bar{v})$ are the first-order terms in the Taylor expansion, of $\bar{N}_0$, $\bar{N}_2$, $\bar{N}_4$. First-order terms related to $\bar{N}_1$ and $\bar{N}_3$ disappear because they are skew-symmetric. They will also appear if the material distribution is not transversally symmetric as may be the case if the thickness is not the same on both sides of the midsurface (for example, stiffened shells, see Fig. 4).

The best first-order variational equation is defined by

$$\text{find } \bar{u} = (\xi_\alpha, \xi_3) \quad \text{in } (H^1_{\gamma_0}(S))^2 \times H^2_{\gamma_0}(S), \tag{5.11}$$

$$A_1(\bar{u},\bar{v}) = \bar{L}(\bar{v}), \quad \bar{v} = (\eta_\alpha, \eta_3) \quad \text{in } (H^1_{\gamma_0}(S))^2 \times H^2_{\gamma_0}(S). \tag{5.12}$$

Theorem 8. The best first order variational eqns (5.11) and (5.12) have a unique solution $\bar{u}_1$. Let $\bar{u}$ be the solution of the full two-dimensional variational eqn (4.25) or (5.1), then there exists a constant $C > 0$ such that

$$\|\bar{u} - \bar{u}_1\| \leqslant C\chi, \quad \chi = h/R. \tag{5.13}$$

Proof. We shall first show the elipticity of eqn (5.12). Let the symmetric tensor $\bar{A}$ be defined by

$$\bar{A}^{\alpha\beta\delta\tau} = \frac{E}{1-\bar{v}^2}\left[\bar{v}a^{\delta\tau}a^{\alpha\beta} + \frac{(1-\bar{v})}{2}(a^{\alpha\delta}a^{\beta\tau} + a^{\alpha\tau}a^{\beta\delta})\right], \tag{5.14}$$

then there exists a constant C such that for every symmetric tensor $\bar{\sigma} = (\sigma_{\alpha\beta})$

$$\bar{A}\bar{\sigma} : \bar{\sigma} \geqslant C\bar{\sigma} : \bar{\sigma} \tag{5.15}$$

Using $\bar{A}$ we have:

$$\begin{aligned} A_1(\bar{u},\bar{u}) &= 2\int_S \bar{A}e{:}\, e\,\mathrm{d}S + \frac{2}{3}\int_S \bar{A}k{:}\, k\,\mathrm{d}S + \frac{4}{3}\int_S \bar{A}e{:}\, Q\,\mathrm{d}S + \frac{2}{5}\int_S \bar{A}Q{:}\,\mathrm{d}S, \\ &= 2\int_S \bar{A}e{:}\, e\,\mathrm{d}S + \frac{2}{3}\int_S \bar{A}k{:}\, k\,\mathrm{d}S + \frac{2}{5}x2\int_S \bar{A}\frac{5}{3}e{:}\, Q\,\mathrm{d}S + \frac{2}{5}\int_S \bar{A}Q{:}\, Q\,\mathrm{d}S. \\ &= 2\int_S \bar{A}e{:}\, e\,\mathrm{d}S + \frac{2}{3}\int_S \bar{A}k{:}\, d\,\mathrm{d}S + \frac{2}{5}\int_S \bar{A}\left(Q + \frac{5}{3}e\right){:}\left(Q + \frac{5}{3}e\right)\mathrm{d}S - \frac{2}{5}\int_S \bar{A}\frac{5}{3}e{:}\,\frac{5}{3}e\,\mathrm{d}S, \\ &\geqslant \frac{8}{9}\int_S \bar{A}e{:}\, e\,\mathrm{d}S + \frac{2}{3}\int_S \bar{A}k{:}\, k\,\mathrm{d}S + \frac{2}{5}\int_S \bar{A}\left(Q + \frac{5}{3}e\right){:}\left(Q + \frac{5}{3}e\right)\mathrm{d}S \\ &\geqslant C\int_S \bar{A}e{:}\, e\,\mathrm{d}S + \int_S \bar{A}k{:}\, k\,\mathrm{d}S \geqslant C(\|e\|^2 + \|k\|^2). \end{aligned}$$

We deduce from Lemma 2 that the coercivity of $A_1(\bar{u},\bar{u})$ and consequently the existence and uniqueness of $\bar{u}_1$.

From eqns (5.1) and (5.12) we deduce by subtraction that for every $\bar{v} = (\eta_\alpha, \eta_3)$ in $(H^1_{\gamma_0}(S))^2 \times H^2_{\gamma_0}(S)$

$$A_1(\bar{u}-\bar{u}_1,\bar{v}) = -zb^{\alpha}_{\tau}B^{\tau}_{\alpha}(\bar{u},\bar{v}).$$

By letting $\bar{v} = \bar{u}-\bar{u}_1$, we deduce from the continuity of $B^{\tau}_{\alpha}(\bar{u},\bar{v})$ and the elipticity of A_1 that

$$\|\bar{u}-\bar{u}_1\|^2 \leqslant C\|\bar{u}\|\,\|\bar{u}-\bar{u}_1\|\chi$$

since $|zb^{\alpha}_{\tau}| \leqslant \chi$ and eqn (5.13) is proved. ◆

Let (u,σ) and (u_1,σ_1) be the displacement and the stress in the shell Ω computed after the full two-dimensional shell eqn (4.25) and the best first order two-dimensional eqn (5.12), respectively, then as a consequent of eqn (5.13) we also have

$$\|u-u_1\| \leqslant C\chi, \tag{5.14}$$

$$\|\sigma-\sigma_1\| \leqslant C\chi. \tag{5.15}$$

These estimations show that it may be sufficient to consider the best first-order two-dimensional shell equations for certain practical cases. It should be observed that the bilinear form A_1 can also be decomposed as $A_1 = A_0+B_0$ where A_0 is the usual bilinear form found in thin shell for the linear Koiter's model. Let u_0 be the solution found in thin shell. Using the same arguments as in Theorem 8 it is also deduced that there exists a constant C such that

$$\|u_1-u_0\| \leqslant C\chi, \quad \text{and} \quad \|\sigma_1-\sigma_0\| \leqslant C\chi. \tag{5.16}$$

It follows immediately that $\|u-u_0\| = 0(\chi)$ and $\|\sigma-\sigma_0\| = 0(\chi)$. These estimations show that the general model found in this framework eqn (4.25) or (4.67) or the best first-order model, eqn (5.12), are suitable for thin shells. Moreover, they provide estimations on the errors committed when the classical model for thin shells is used.

If necessary, instead of the full equation, a best second- or third-order variational equation may rather be used by considering additional terms in eqn (5.1) and this will lead to $\|u-u_p\| = 0(\chi^p)$, and $\|\sigma-\sigma_p\| = 0(\chi^p)$, $p = 2, 3, \ldots$ These considerations will depend on the type of practical problems to be solved.

We shall now present the de-scaled two-dimensional boundary value equations by using the de-scaling relations on the transverse displacement ξ^{ε}_3, the in-plane displacements $\xi^{\varepsilon}_{\alpha}$ and the curvature tensor $(b^{\varepsilon})^{\alpha}_{\beta}$:

$$\xi^{\varepsilon}_{\alpha}(x^1,x^2) = \varepsilon^2\xi_{\alpha}(x^1,x^2); \quad \xi^{\varepsilon}_3(x^1,x^2) = \varepsilon\xi_3(x^1,x^2) \quad \text{for all } (x^1,x^2) \quad \text{in } \bar{S} \tag{5.17}$$

$$b^{\varepsilon}_{\alpha\beta} = \varepsilon b_{\alpha\beta}, \quad (b^{\varepsilon})^{\alpha}_{\beta} = \frac{1}{\varepsilon}(b)^{\alpha}_{\beta}, \quad z^{\varepsilon} = \varepsilon z, \quad z = x^3; \tag{5.18}$$

the shell displacement consequently satisfies the relations:

$$u^{\varepsilon}_{\alpha}(x^{\varepsilon}) = \varepsilon^2 u_{\alpha}(0)(x); \quad u^{\varepsilon}_3(x^{\varepsilon}) = \varepsilon u_3(0)(x) = \varepsilon\xi_3(x^1,x^2).$$

Let $\mathbf{t}$, $\boldsymbol{v}$ be the unit tangent and outer normal vectors, respectively, on the border γ_1 of S. Let $\boldsymbol{v}$, $\mathbf{t}$, a^3 form a direct orthonormal basis in IR^3 and $\bar{v} = (\eta_{\alpha},\eta_3)$ in $(H^1_{\gamma_0}(S))^2 \times H^2_{\gamma_0}(S)$. We define

$$n_v = -\boldsymbol{v}\cdot(b^{\beta}_{\alpha}\eta_{\beta}+\nabla_{\alpha}\eta_3)a^{\alpha}, \quad n_t = -\mathbf{t}\cdot(b^{\beta}_{\alpha}\eta_{\beta}+\nabla_{\alpha}\eta_3)a^{\alpha};$$

then

$$\int_{\gamma_1} m^\alpha \theta_\alpha \, d\gamma = \int_{\gamma_1} (m^s n_v + m^v n_t) \, d\gamma, \quad m^s = m \cdot \mathbf{v}, \quad ,m^v = m \cdot \mathbf{t}$$

where m^s and m^v are flexural and torsional moment density, respectively, on the border γ_1 of S. We shall denote by m^s_ε, m^v_ε, p^α_ε, p^3_ε, q^α_ε, q^3_ε the homologous of m^s, m^v, p^α, p^3, q^α, q^3 obtained after de-scaling by replacing all variables X found in the integrals by their homologous X^ε and integrating from $-\varepsilon$ to ε.

Let $\bar{u}^\varepsilon = (\xi^\varepsilon_\alpha, \xi^\varepsilon_3)$

$$N^{\alpha\beta}_\varepsilon = \frac{2\varepsilon E^\varepsilon}{1-\bar{v}^2}[(1-\bar{v})e^{\alpha\beta}(\bar{u}^\varepsilon) + \bar{v}e^\rho_\rho(\bar{u}^\varepsilon)a^{\alpha\beta}], \tag{5.19}$$

$$M^{\alpha\beta}_\varepsilon = \frac{2\varepsilon^3 E^\varepsilon}{3(1-v^2)}[(1-\bar{v})k^{\alpha\beta}(\bar{u}^\varepsilon) + \bar{v}k^\rho_\rho(\bar{u}^\varepsilon)a^{\alpha\beta}], \tag{5.20}$$

$$\bar{\boldsymbol{M}}^{\alpha\beta}_\varepsilon = \frac{2\boldsymbol{\varepsilon}^3\boldsymbol{E}^\varepsilon}{3(1-\bar{\boldsymbol{v}}^2)}[(1-\bar{\boldsymbol{v}})\boldsymbol{e}^{\alpha\beta}(\bar{\boldsymbol{u}}^\varepsilon) + \bar{\boldsymbol{v}}\boldsymbol{e}^\rho_\rho(\bar{\boldsymbol{u}}^\varepsilon)\boldsymbol{a}^{\alpha\beta}], \tag{5.21}$$

$$\bar{\boldsymbol{N}}^{\alpha\beta}_\varepsilon = \frac{2\boldsymbol{\varepsilon}^3\boldsymbol{E}^\varepsilon}{3(1-\bar{\boldsymbol{v}}^2)}[(1-\bar{\boldsymbol{v}})\boldsymbol{Q}^{\alpha\beta}(\bar{\boldsymbol{u}}^\varepsilon) + \bar{\boldsymbol{v}}\boldsymbol{Q}^\rho_\rho(\bar{\boldsymbol{u}}^\varepsilon(\boldsymbol{a}^{\alpha\beta}], \tag{5.22}$$

$$\bar{\bar{\boldsymbol{M}}}^{\alpha\beta}_\varepsilon = \frac{2\boldsymbol{\varepsilon}^5\boldsymbol{E}^\varepsilon}{5(1-\bar{\boldsymbol{v}}^2)}[(1-\bar{\boldsymbol{v}})\boldsymbol{Q}^{\alpha\beta}(\bar{\boldsymbol{u}}^\varepsilon) + \bar{\boldsymbol{v}}\boldsymbol{Q}^\rho_\rho(\bar{\boldsymbol{u}}^\varepsilon)\boldsymbol{a}^{\alpha\beta}]. \tag{5.23}$$

The best first-order two-dimensional de-scaled shell variational equation is:

$$\int_S (N_\varepsilon : e(\bar{v}) + M_\varepsilon : k(\bar{v}) + \bar{N}_\varepsilon : e(\bar{v}) + \bar{M}_\varepsilon : Q(\bar{v}) + \bar{\bar{M}}_\varepsilon : Q(\bar{\beta})) \, dS$$

$$= \int_S [p^\alpha_\varepsilon \eta_\alpha + p^3_\varepsilon \eta_3] \, ds + \int_{\gamma_1} [q^\alpha_\varepsilon \eta_\alpha + q^3_\varepsilon \eta_3] \, ds + \int_{\gamma_1} (m^s_\varepsilon n_v + m^v_\varepsilon n_t) \, d\gamma;$$

$$\bar{v} = (\eta_\alpha, \eta_3) \quad \text{is in } (H^1_{\gamma_0}(S))^2 \times H^2_{\gamma_0}(S). \tag{5.24}$$

The first two terms on the left are those found in thin shells. The upperscored membrane stress tensor $\bar{N}$, the upperscored and double upperscored flexural moment stress tensors $\bar{M}$ and $\bar{\bar{M}}$ (respectively) are new terms which modify the usual in-plane stress and moment. From their definitions we deduce that:

the membrane stress is $N_\varepsilon + \bar{N}_\varepsilon$ while the moment is $M_\varepsilon + \bar{M}_\varepsilon + \bar{\bar{M}}_\varepsilon$.

In most computer aided design programs steel reinforcements are computed by using N_ε and M_ε. One will have to replace N_ε by $N_\varepsilon + \bar{N}_\varepsilon$ and M_ε by $M_\varepsilon + \bar{M}_\varepsilon + \bar{\bar{M}}_\varepsilon$ (Capra and Maury, 1978).

Let us recall the two-dimensional covariant version of Green's formula:

$$\int_S \nabla_\alpha M^{\alpha\beta}\,\mathrm{d}S = \int_{\partial S} M^{\alpha\beta} v_\alpha\,\mathrm{d}\gamma \quad \text{or} \quad \int_S \nabla_\alpha M^{\alpha\beta} v_\beta\,\mathrm{d}S = -\int_S M^{\alpha\beta}\nabla_\alpha v_\beta\,\mathrm{d}S + \int_{\partial S} M^{\alpha\beta} v_\alpha v_\beta\,\mathrm{d}\gamma.$$

We shall denote by

$$b_\tau^v = b_{\tau\alpha} v^\alpha, \quad b_\tau^\tau = b_{\tau\alpha} t^\alpha, \quad M^{v\tau} = M^{\alpha\beta} v_\alpha t_\beta, \quad \partial_s = \partial_{\mathbf{t}}, \quad M^{v\tau} = M^{\alpha\tau} v_\alpha, \quad \theta_\alpha = -(b_\alpha^\gamma \eta_\gamma + \nabla_\alpha \eta_3).$$

From the variational equation and by using Green's formula and the above definitions, proceeding as Bamberger (1975), we derive the following boundary value problem:

on $\boldsymbol{S}$:

$$p^\beta + \nabla_\alpha N^{\alpha\beta} + 2(\nabla_\alpha M^{\alpha\tau}) b_\tau^\beta + M^{\alpha\tau}(\nabla_\tau b_\alpha^\beta) + \boldsymbol{\nabla_\alpha(\bar{M}^{\gamma\alpha} b_\gamma^\tau) b_\tau^\beta + \nabla_\alpha \bar{N}^{\alpha\beta} + \nabla_\alpha(\bar{\bar{M}}^{\gamma\alpha} b_\gamma^\tau) b_\tau^\beta} = 0, \tag{5.25}$$

$$-p^3 + \nabla_\alpha\nabla_\beta M^{\alpha\beta} - N^{\alpha\beta} b_{\alpha\beta} - b_\tau^\beta b_{\alpha\beta} M^{\alpha\tau} + \boldsymbol{\nabla_\alpha\nabla_\beta(\bar{M}^{\tau\beta} b_\tau^\alpha) - \bar{N}^{\alpha\beta} b_{\alpha\beta}} + \nabla_\alpha\boldsymbol{\nabla_\beta(\bar{\bar{M}}^{\tau\beta} b_\tau^\alpha} = 0, \tag{5.26}$$

on the free border γ_1:

$$N^{v\beta} + M^{v\tau} b_\tau^\beta + (M^{v\tau} - m^v) b_\tau^\beta - q^\beta + \boldsymbol{\bar{M}^{v\tau} b_\tau^{\mathbf{t}} b_{\mathbf{t}}^\beta + \bar{N}^{v\beta} + \bar{\bar{M}}^{v\tau} b_\tau^{\mathbf{t}} b_{\mathbf{t}}^\beta} = 0, \tag{5.27}$$

$$\nabla_\beta M^{\alpha\beta} v_\alpha - \partial_s(m^v - M^{vt}) + q + \boldsymbol{\nabla_\beta(\bar{M}^{\alpha\tau} b_\tau^\beta) v_\alpha + \nabla_\beta(\bar{\bar{M}}^{\alpha\tau} b_\tau^\beta) v_\alpha} + \partial_{\mathbf{s}}\boldsymbol{(\bar{M}^{v\tau} b_\tau^{\mathbf{t}} + \bar{\bar{M}}^{v\tau} b_\tau^{\mathbf{t}})} = 0, \tag{5.28}$$

$$M^{vv} + m^s + \boldsymbol{\bar{M}^{v\tau} b_\tau^v + \bar{\bar{M}}^{v\tau} b_\tau^v} = 0, \tag{5.29}$$

on the clamped border γ_0:

$$\xi_i = 0, \quad \partial_v \xi_3 = 0. \tag{5.30}$$

We have voluntarily omitted the subscripts ε on $b_{\alpha\beta}^\varepsilon$ and $(b^\varepsilon)_\alpha^\beta$. The real displacements in the shell is

$$u_\alpha^\varepsilon(x^1, x^2, z^\varepsilon) = \xi_\alpha^\varepsilon - z^\varepsilon(2b_\alpha^{\varepsilon\tau}\xi_\tau^\varepsilon + \nabla_\alpha\xi_3^\varepsilon) + (z^\varepsilon)^2(b_\alpha^{\varepsilon\tau} b_\tau^{\varepsilon\gamma}\xi_\gamma^\varepsilon + b_\alpha^{\varepsilon\tau}\nabla_\tau\xi_3^\varepsilon), \tag{5.31}$$

$$u_3^\varepsilon(x^1, x^2, z^3) = \xi_3^\varepsilon; \tag{5.32}$$

and the stresses $\sigma^{\alpha\beta}(x^1, x^2, z^\varepsilon)$ are given by

$$\sigma_\varepsilon^{\alpha\beta}(x^1, x^2, z^\varepsilon) = \frac{1}{2\varepsilon} N_\varepsilon^{\alpha\beta} - \frac{3z^\varepsilon}{2\varepsilon^3} M_\varepsilon^{\alpha\beta} + \boldsymbol{\frac{5(z^\varepsilon)^2}{2\varepsilon^5}\bar{\bar{M}}_\varepsilon^{\alpha\beta}}; \tag{5.33}$$

while the shear stresses $\sigma_\varepsilon^{\alpha 3}(x^1, x^2, z^\varepsilon)$, $\sigma_\varepsilon^{33}(x^1, x^2, z^\varepsilon)$ are computed by solving eqns (4.47) and (4.48) and (4.51) and (4.52) in which the subscript ε is added to homologous terms.

The upperscored terms are non classical terms while the others are those found in engineering literature. From their definitions these new terms are related to the third fundamental from. Their energy contribution can then be estimated and will thus enable engineers to choose one model or the other. We recall again that the form of the variational eqn (5.24) is due to the transverse symmetric distribution of material across the midsurface and so depend on the form of the in-plane stresses.

6. Discussion and comments

We first recall that the geometry of the shell is general. We only imposed the condition that $\chi < 1$ (which implies that $\rho > 0$ eqn (2.5)) and the local chart be at least $W^{2,\infty}$ (which means that C^1 midsurface are considered because of the compact embedding of $W^{2,\infty}$ in C^1). The usual classical condition C^3 is consequently treated. No uniform elipticity condition is imposed on the midsurface. The two-dimensional model of the real shell eqn (4.67) (which is the main objective of this paper), obtained from eqn (4.25) through the de-scaling relations and integrations was deduced from the three-dimensional problem and strong convergence results have been proved. No a priori assumptions whether of a geometrical or mechanical nature were made. The fact that the modulus tensor on the midsurface depends on the thickness and the curvature is in fact natural. Clearly two clamped spherical slabs, one convex, the other concave do not offer the same resistance to the same transversal loading. The estimations in Section 5 show that this model can be used for thin shells, membrane or flexural shells as well as thick shells.

It should be recalled that in our approach the small parameter is half the thickness h. The parameter $\chi = h/R$ found in earlier mechanics appears naturally because the full expression of the contravariant basis vector g^α is used. Because of this, the parameter χ appears in the strain tensor, eqn (4.18), in the modulus tensor, eqn (4.66) and consequently in eqn (4.67).

The main difference between our final displacement and that found in the literature is due to the following reason. In the classical framework used in thin shell, the basis vectors of the unscaled shell Ω^ε are still assumed to be basis vectors of the scaled shell Ω. This hypothesis induces an approximation on the scaled metric. Moreover the contravariant basis vector g^α_ε is approximated and in the limit, is equal to $\bar{g}^\alpha = (\delta^\alpha_\beta + z b^\alpha_\beta) a^\beta$ in the scaled shell.

It should also be noted that this is exactly the first order approximation of $(\mu^{-1})^\alpha_\beta a^\beta$ (see Lemma 1). The limit displacement obtained $\bar{u}_\alpha = \xi_\alpha - z\nabla_\alpha \xi_3$, $\bar{u}_3 = \xi_3$ (which is the classical Kirchhoff–Love displacement) is expressed in the basis $\{a^\alpha, a^3\}$ instead of the basis $\{\bar{g}^\alpha, a^3\}$. It should be noted that this Kirchhoff–Love displacement satisfies the equations $\epsilon_{i3}(u) = 0$, in which the expressions of the shear strain $\epsilon_{i3}(u)$ have been approximated. By neglecting certain terms of the scalar product $(I + \nabla u)A_3 \cdot a_\alpha$ (A_3 is a normal vector to the midsurface and a_α is a tangent basis vector of the deformed midsurface) one obtains $(I + \nabla u)A_3 \cdot a_\alpha \approx 0$ and the classical assert: that a Kirchhoff–Love displacement transforms normals to the midsurface to normals to the deformed midsurface is thus deduced.

Because of these same approximations it is also said that a Reissner–Mindlin displacement $u_\alpha = \xi_\alpha - z\theta_\alpha(x^1, x^2)$, $u_3 = \xi_3$ where ξ_i, θ_α are five unknown functions is not a Kirchhoff–Love displacement. In this framework, the exact expressions of $\epsilon_{i3}(u)$ are used; it appears that both the classical Kirchhoff–Love and Reissner–Mindlin displacements are linear approximations of the final Kirchhoff–Love displacement we have obtained. In fact the exact transverse shear strains of the classical Kirchhoff–Love displacement are $\varepsilon_{\alpha 3}(u) = b^\tau_\alpha \bar{u}_{\tau/2} \neq 0$. Such a displacement is therefore inadmissible because it is shear strain free in thick shells (the resultant deformation is not planar). Moreover $Q_{\alpha\beta}(\bar{u}) \neq 0$. By using thin shell model, the following term in the deformation energy

$$(1/2) \int_S (\bar{N}^\varepsilon_2 e\colon Q + \bar{N}^\varepsilon_2 Q\colon e + \bar{N}^\varepsilon_4 Q\colon Q - \bar{N}^\varepsilon_1 e\colon k - \bar{N}^\varepsilon_1 k\colon e - \bar{N}^\varepsilon_3 k\colon Q - \bar{N}^\varepsilon_3 Q\colon k)\,\mathrm{d}S$$

disappears and by applying finite element methods, stiffness matrix obtained in this way will obviously be ill-conditioned.

The Reissner–Mindlin displacement gives a better result in computational mechanics when a plate is 'moderately thick' probably because it is a better linear approximation of the displacement of a thick shell.

The new terms that appear in our model and the correction that naturally appears on the strains may probably ameliorate 'the locking phenomenon' encountered in numerical computation, because their energy contribution will modify stiffness matrix (Chapelle, 1995).

Clearly the limit displacement found in our approach which is a Kirchhoff–Love displacement (yields planar deformation) does not transform normals to the midsurface to normals to the deformed midsurface. So also the Navier–Bernoulli displacements on beams or rods are reviewed in this framework.

It is well known in practice that cross sections of beams with great height do not satisfy the Navier–Bernoulli displacement principle.

Though the Reissner–Mindlin displacement does not transform normals to the midsurface to normals to the deformed midsurface, Reissner's approach is different. It is based on the geometrical assumption that the three-dimensional displacement is of the form stated, but this assertion is yet to be mathematically substantiated.

Another theory which provides non classical Kirchhoff–Love displacement consists of imposing a formal power series expansion of the three-dimensional solution and to construct the successive terms (Goldenveizer, 1963, 1964). In this approach some a priori restrictive mechanical assumptions are imposed on the stress and strain distribution across the midsurface. Some difficulties arise on the boundary conditions. Convergence results are still to be improved.

Naghdi's approach imposes at the beginning of the analysis a geometrical assumption on the type of displacement that the shell undergoes (displacement with five unknown functions) and a mechanical assumption on the stress field (planar stress distribution). Such assumptions have not been deduced from the three-dimensional model.

In Section 5 we emphasized on the best first-order approximation of the full variational eqn (4.67) because the surface rigidities may be expensive in computation. In some practical cases because of the error estimate established, only an n-th approximation of these rigidities may be sufficient. Unlike in the power series expansion method, the n-th approximate solution which in our approach is the solution of eqn (4.67) with approximate surface rigidities is still of the form of eqns (4.16) and (4.17) and therefore satisfies boundary conditions.

Our limit displacement can be written as

$$u_\alpha(x^1,x^2,z) = \xi_\alpha(x^1,x^2)+z\theta_\alpha(x^1,x^2)+z^2\psi_\alpha(x^1,x^2), \quad u_3(x^1,x^2,z) = \xi_3(x^1,x^2).$$

This approach is appropriate to study torsional loading and will probably yield better results if the seven functions are considered as independent unknowns. Such displacements are shear strain free. It will be much easier to implement numerically since it will only require finite elements of class C^0 (Ciarlet, 1978).

A general displacement for shells of the form $u_i(x^1,x^2,z) = \xi_i(x^1,x^2)+z\theta_i(x^1,x^2)+z^2\psi_i(x^1,x^2)$ yields different results probably more realistic as it preserves the shear strain and transversal deformation. Details on this approach will be presented in Nzengwa (1998b).

Another important aspect of this model is that it ca be applied to stiffened shells. In reality a stiffened shell can neither be considered as a thin shell nor a 'moderately thick shell' because in certain cases the stiffeners may be very thick locally and it may be impossible to mesh such a three-dimensional structure if finite element method is used for computation (Fig. 4). Our model will still be applied provided $\chi = h/R < 1$. If the midsurface is not stiffened symmetrically then the shell can be immersed in a symmetric stiffened shell (obtained by reflection over the midsurface for example) and the rigidities (4.66b) will be integrated in the real shell.

In certain computer programs widely used, the stiffeners are considered as beam elements while the rest of the shell is assumed to be thin. Particular finite elements are used at the junctions between the shell (rest of the shell) and the stiffeners (Combescure, 1994). Though numerical computations have been performed, no mathematical justifications have been provided. Such complex finite elements are not necessary in our approach. It can be deduced from our displacement field that no part of a cross section of the shell undergoes beam displacement. In stiffened shells, even if the best first order two-dimensional model is used, the final shell equation will be non-homogeneous. In most practical cases the midsurface distribution of the rigidities will be periodic or even almost periodic and may vary rapidly. Homogenizing rigidities so obtained is of great practical interest. These aspects have also been analyzed by the authors.

The authors have also analyzed the effect of this model in the elastic dynamic behaviour of thick shells and the non-linear elastic model for thick plates. This approach will be extended to the elasto-plastic analysis and the junctions of shells in a rather inherent slightly different approach of that found in Ciarlet et al. (1989).

Acknowledgement

The authors thank the referees for their kind advice.

References

Adams, R.A., 1975. Sobolev Spaces. Academic Press, New York.

Bamberger, Y., 1975. Cours de Plaques et Coques, Polycopié de l'Ecole Nationale des Ponts et Chaussées. ENPC, Paris.

Bernadou, M., 1994. Méthodes d'Elements Finis pour les Coques Minces. Masson, Paris.

Bernadou, M., Ciarlet, P.G., 1976. Sur l'ellipticité du modèle linéaire de coques de W.T. Koiter. Glowinsky, R., Lions, J.L. (Eds.), Computing Methods in Sciences and Engineering. Lecture Notes in Economics and Systems, vol. 134. Springer-Verlag, Berlin, pp. 89–136.

Blouza, A., Le Dret, H., 1995. Existence and uniqueness for the linear Koiter model for shells with little regularity. Comptes Rendus de l'Académie des Sciences (CRAS) Paris, Série I, 317, 327–329.

Brezzi, F., Fortin, M., 1986. Numerical approximation of Midlin–Reissner plates. Math. Comp. 47, 151–158.

Brezis, H., 1973. Opérateurs Maximaux Monotones et Semi-Groupes de Contractions dans les Espaces de Hilbert. North-Holland, Amsterdam.

Brezis, H., 1983. Analyse fonctionnelle. Théorie et Application. Masson, Paris.

Budiansky, B., Sanders, J.L., 1967. On the best first order linear shell theory. Prog. Appl. Mech. W. Prager Anniversary Volume. Macmillan, New York, pp. 129–140.

Capra, A., Maury, J.F., 1978. Calcul automatique du ferraillage optimal des Plaques ou Coques en Béton Armé. Annales de l'Institut Technique du Batiment et des Travaux Publics, Serie Informatique Appliquée, No. 367 Decembre.

Chapelle, D., 1995. A locking-free approximation of curved rods by straight beam elements, to appear in Numerische Mathematik.

Ciarlet, P.G., 1978. The Finite Element Method for Elliptic Problems. North-Holland, Amsterdam.

Ciarlet, P.G., 1980. A justification of the Von Karman equation. Arch. Rat. Mech. Anal. 73, 349–389.

Ciarlet, P.G., 1988. Mathematical Elasticity, vol. I, Three-dimensional Elasticity. North-Holland, Amsterdam.

Ciarlet, P.G., 1990. Plates and Junctions in Elastic Multi-Structures. An Asymptotic Analysis. Masson, Paris and Springer-Verlag, Heilderberg.

Ciarlet, P.G., Lods, V., 1994. Analyse asymptotique des coques linéairement élastiques; I Coques 'membranaires'. CRAS Paris, Série I, 318, 863–868.

Ciarlet, P.G., Lods, V., 1994. Analyse asymptotique des coques linéairement élastiques; II Coques 'en flexion'. Comptes Rendus de l'Académie des Sciences CRAS Paris, Série I, 319, 95–100.

Ciarlet, P.G., Lods, V., 1994. Analyse asymptotique des coques linéairement élastiques; III une justification du modèle de W.T. Koiter. CRAS Paris, Série I, 319, 299–304.

Ciarlet, P.G., Lods, V., 1994. Analyse asymptotique des coques linéairement élastiques; IV Coques 'membranaires sensitives'. CRAS Paris, Série I, 321, 649–654.

Ciarlet, P.G., Lods, V., 1996. Asymptotic Analysis of Linearly Elastic Shells. I. Justification of Membrane Shell Equations. Arch. Rational Mech. Anal. 136, Springer-Verlag, pp. 119–161.

Ciarlet, P.G., Lods, V., 1996. Asymptotic Analysis of Linearly Elastic Shells. III. Justification of Koiter's Shell Equations. Arch. Rational Mech. Anal. 136, Springer-Verlag, pp. 191–200.

Ciarlet, P.G., Lods, V., 1996. Asymptotic analysis of linearly elastic shells: generalized membrane shells. Journal of Elasticity 43, 147–188. Kluwer Academic Publishers. Printed in the Netherlands.

Ciarlet, P.G., Lods, V., 1996. On the elipticity of linear membrane shell equations. J. Math. Pures Appl. 75, 107–124.

Ciarlet, P.G., Miara, B., 1992. Justification of the Two-Dimensional Equations of a Linearly Elastic Shallow Shell. Comm. Pure and Appl. Math. vol. XLV. John Wiley and Sons Inc., pp. 327–360.

Ciarlet, P.G., Le Dret, H., Nzengwa, R., 1987. Modélisation de la junction entre un corps tridimensionnel et une plaque. Comptes Rendus de l'Académie des Sciences CRAS Paris, Série I, 305.

Ciarlet, P.G., Le Dret, H., Nzengwa, R., 1989. Junctions between three-dimensional and two-dimensional linearly elastic structures. J. Math. Pures et Appl. 68, 261–295.

Ciarlet, P.G., Lods, V., Miara, B., 1996. Asymptotic analysis of linearly elastic shells. II. Justification of Flexural Shell Equations. Arch. Rational Mech. Anal. 136. Springer-Verlag, pp. 163–190.

Combescure, A., 1995. Les Travaux d'Habilitations à Diriger les Recherche (HDR) No. 94012. Laboratoire de Mécanique et Technologie (LMT), Ecole Normale Supérieure de Cachan.

Crouzeix, M., Mignot, A.L., 1983. Analyse Numérique des Équations Différentielles. Masson, Paris.

Destuynder, P., 1980. Sur une justification des modèles des plaques et coques par les méthodes asymptotiques. Ph.D. dissertation, Université Pierre et Marie Curie Paris 6.

Destuynder, P., 1986. Une Théorie Asymptotique des Plaques Minces en Élasticité Linéaire. Masson, Paris.

Do Carmo, M., 1976. Differential Geometry of Curves and Surfaces. Prentice-Hall.

Germain, P., Muller, P., 1986. Introduction à la Mécanique des Milieux Continus. Masson, Paris.

Goldenveizer, A.L., 1963. Derivation of an approximate theory of shells by means of asymptotic integration of the equations of the theory of elasticity. Prikl. Mat. Mech. 27, 593–608.

Goldenveizer, A.L., 1964. The principles of reducing three-dimensional problems in elasticity to two-dimensional problems of the theory of plates and shells. In: Görtler, H. (Ed.), Proceedings of the 11th International Congress of Theoretical and Applied Mechanics. Springer-Verlag, Berlin, pp. 306–311.

Green, A.E., Zerna, W., 1968. Theoretical Elasticity. University Press, Oxford.

Greub, W.H., 1969. Linear Algebra. Springer-Verlag, Berlin.

John, F., 1965. Estimates for the derivatives of the stresses in a thin shell and interior shell equation. Comm. Pure Appl. Math. 18, 235–267.

John, F., 1971. Refined interior equations for thin elastic shells. Comm. Pure Appl. Math. 24, 583–615.

Kirchhoff, G., 1850. Über das Gleichgewicht und die Bewegung einer elastischen Scheibe. J. Reine Angew. Math. 40, 51–58.

Klingenberg, W., 1982. Riemannian Geometry. de Gruyter.

Koiter, W.T., 1970. On the foundation of the linear theory of thin elastic shells, Parts I and II, roc. Kon. Ned. Akad. Wetensch. B73, 169–195.

Le Dret, H., 1991. Problèmes Variationnels dans les Multi-Domaines: Modélisation des Jonctions et Applications. Masson, Paris.

Lelong, P., Ferrand, J., 1963. Géometrie Différentielle. Masson, Paris.

Lions, J.L., Magenes, E., 1968. Problèmes aux Limits non Homogènes et Applications, vol. 1. Dunod, Paris.
Love, A.E.H., 1934. A Treatise on the Mathematical Theory of Elasticity, 4th ed. Cambridge University Press, Cambridge.
Mindlin, R.D., 1951. Influence of rotatory inertia and shear on flexural motions of isotropic elastic plates. J. Appl. Mech. 18, 31–38.
Morgenstern, D., 1959. Herleitung der Plattentheorie aus der dreidimensionalen Elastizitätheorie. Arch. Rational Mech. Anal, 4, 145–152.
Naghdi, P.M., 1970. The theory of shells and plates. In: Flugge, S., Truesdell, C., (Eds.), Hanbuch der Physik, vol. VI a/2, pp. 425–640, Berlin.
Nering, I., 1970. Theory of Thin Shells. Springer-Verlag, berlin.
Niordson, F.I., 1969. Theory of Thin Shells. Springer-Verlag, Berlin.
Nguetseng, G., Nzengwa, R., Tagne Simo, B.H., 1995. A homogenized two-dimensional model for almost periodic thick shells, submitted for publication.
Nguetseng, G., Nzengwa, R., Tagne Simo, B.H., 1995. A homogenized two-dimensional model for non-homogeneous periodic thick shells, Part I and II, to appear.
Nordgreen, R.P., 1972. A bound on the error in Reissner's theory of plate. Quart. Appl. Math. 551–556.
Nzengwa, R., 1998a. Mécanique des Milieux Continus. Cours de l'Ecole Nationale Supérieure Polytechnique de Yaoundé, submitted for publication.
Nzengwa, R., 1998b. Elastic shell theory. Cours de l'Ecole Nationale Supérieure Polytechnique de Yaoundé, submitted for publication.
Nzengwa, R., Tagne Simo, B.H., 1994. A limit analysis of the local theory for linear elastic shells. Giraga V, Cagnac, F., Wouafo Kamga, J., Ezin, J.P., (Eds.), Côme Goudjo, Yaoundé, 13–24 Septembre 1994, pp. 318–324.
Nzengwa, R., Tagne Simo, B.H., 1995. A two-dimensional model for nonlinearly elastic thick shells, submitted for publication.
Nzengwa, R., Tagne Simo, B.H., 1995. A dynamic analysis for elastic thick shells, submitted for publication.
Reissner, E., 1985. Reflections on the theory of elastic plates. Appl. Mech. Rev. 38, 1453–1464.
Rougee, P., 1969. Equilibre des coques élastiques minces inhomogènes en théorie non linéaire. Ph.D. dissertation, Université Paris.
Schwartz, L., 1996. Théorie des Distributions. Hermann, Paris.
Spivak, M., 1975. Differential Geometry, vols. 3, 4, 5, Publish of Perish.
Timoshenko, S., Woinowsky-Kreiger, W., 1958. Theory of Plates and Shells. McGraw-Hill, New York.